TECHNICAL ASSISTANCE REPORT

SOUTH AFRICA

Stress Testing the Central Bank Balance Sheet and Calibrating Risk-based Capital Buffers

May 2025

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Glossary

CBST	Central Bank Stress Test
CiC	Currency in Circulation
GBP	British Pound
GFECRA	Gold and Foreign Exchange Contingency Reserve Account
MCM	Monetary and Capital Markets Department
NT	National Treasury
TA	Technical Assistance
SARB	South African Reserve Bank
USD	U.S. Dollar
VaR	Value-at-Risk
ZAR	South African Rand

Preface

At the request of the South African Reserve Bank, a Monetary and Capital Markets (MCM) Department mission visited Pretoria, South Africa, from July 17–26, 2024, to assist the authorities in stress testing their balance sheet and to calibrate risk-based capital buffers. The mission was a hybrid one, with Darryl King and Yuji Sakurai attending in person, and Luyao Liu participating virtually.

The mission met with Deputy Governors, Rashad Cassim and Mampho Modise; Advisor to the Governors, David Fowkes; Chief Operating Officer, Pradeep Maharaj; and staff from the Financial Services, Legal, Economic Research, Financial Markets, and Financial Stability Departments. The mission wishes to thank staff, and particularly David Fowkes, for their cooperation, productive discussions, and hospitality.

Executive Summary

A Monetary and Capital Markets (MCM) Department mission in December 2023 assessed the South African Reserve Bank's (SARB) financial arrangements, with a focus on the Gold and Foreign Exchange Contingency Reserve Account (GFECRA). That mission emphasized that the financial arrangements should be grounded in the SARB's legal framework and underpin its policy solvency—a central bank can be considered "policy solvent" if, over time, it has realized earnings greater than its monetary policy and operating costs. Several recommendations were made (Table 1), including for the SARB and the National Treasury (NT) to establish a high-level framework for distributing amounts accrued in GFECRA, establishing a risk-based buffer for the contingency reserve (i.e., equity), and for the SARB and the NT to publish a memorandum of understanding to ensure transparency around the financial arrangements.

Since that mission, and as part of the February 2024 Budget, the SARB and the NT announced the following principles underpinning a new financial framework:

- The SARB's policy solvency should not be undermined by any GFECRA distribution.
- There should be no sales of foreign exchange (FX) to realize GFECRA gains if such reserves are below estimated adequacy levels.
- There should be no distribution of unrealized GFECRA balances that could plausibly be unwound by future rand appreciations.
- GFECRA distributions will be used to reduce government borrowing.
- Any GFECRA distributions should be governed by a framework that rules out ad hoc decisions.
- GFECRA settlement arrangements should be public to ensure transparency.

The framework involves a "waterfall" with buffers calibrated for the GFECRA and Contingency Reserve. The GFECRA buffer is calibrated to be able to absorb large and plausible rand appreciations. Once this buffer is met, surplus balances are transferred to the Contingency Reserve, which is also set with a buffer. This buffer is calibrated to meet possible losses arising from SARB activities, including monetary policy costs (i.e., interest on excess reserves). Once this buffer is satisfied, surplus funds are then transferred to the NT. Based on a preliminary analysis, the SARB and the NT agreed that the GFECRA buffer for the first year would be set at ZAR 250 billion, allowing for distribution of ZAR 250 billion. Of this, a net transfer to the NT of ZAR 150 billion was agreed upon, with ZAR 250 billion paid by the SARB to the NT and ZAR 100 billion transferred to the SARB from the NT, to reduce government borrowing. ZAR 100 billion was transferred to the SARB's Contingency Reserve. No explicit target for this buffer was set.[1]

The mission demonstrated a Central Bank Stress Testing (CBST) model to enable the SARB to institutionalize a process that forecasts its balance sheet based on macroeconomic scenarios. The model incorporates accounting rules and equations derived from macroeconomics and finance. The model generates total equity decomposed into realized earnings and the revaluation account. Satellite models are developed for forecasting several key balance sheet items (e.g., currency in circulation (CiC)) given the macroeconomic scenarios. Two macro-scenarios are used to examine the SARB's balance

[1] The transfers of 100 billion ZAR to SARB's contingency reserve and the first 100 billion ZAR to the National Treasury are in the fiscal year of 2024/2025, while the initial balance sheet is based on the financial statement ending on 2024/3/31 in our central balance sheet stress testing. This transfer to the contingency reserve was intended to promote the policy solvency of the SARB.

sheet. In the base case, GDP growth is expected to be low but stable and accompanied by mild inflation. In the adverse case, external shocks lead to a temporary negative macroeconomic shock with an assumed recovery after 2026. The SARB balance sheet is robust because it has relatively few interest-bearing liabilities. It has positive net foreign assets and positive net claims on the government.

Central to the new framework is the calibration of the GFECRA buffer to ensure distributions would not plausibly need to be reversed due to future rand appreciations. The mission recommended a value-at-risk (VaR) approach to be applied to FX revaluation losses. The VaR is the threshold that determines the worst loss the account can absorb, within the distribution of possible outcomes. Here, the SARB needs to determine two risk tolerance parameters: the percentile (i.e., how far into the tail) and the time horizon. The mission provides the analysis that assists SARB in determining these risk tolerance parameters—the current calibration of the GFECRA buffer (ZAR 250 billion) approximately corresponds to the 5th percentile with 2-year time horizon (243 billion ZAR). This calibration would seem to be both conservative and appropriate.

The adequacy of the current equity buffer (142 billion ZAR) is assessed by using the combination of the FX-at-Risk and Inflation-at-Risk scenarios. The combination of these two scenarios is selected to effectively stress the SARB balance sheet. On the asset side, ZAR appreciation lowers foreign interest income. On the liability side, higher inflation increases the domestic interest expenses on the excess reserves. A combination of these two scenarios—lower income and more expenses—depletes equity more quickly. Equity (ZAR 142 billion after the budgeted ZAR 100 billion transfer) is forecasted to remain positive over the next 13 years under the severest combination of FX-at-Risk and Inflation-at-Risk. The approach toward assessing this buffer is somewhat different from that of the GFECRA since it is computationally challenging to derive the buffer based on identified risk tolerance (i.e., VaR), although it can be done. It is noted that the required buffer increases for a given risk tolerance if foreign reserves are accumulated. Also, no account is taken of the potential credit risks associated with the SARB's lender of last resort function.

Given the SARB's public commitment to annually recalibrate the two buffers, it is paramount that it institutionalizes and publicly communicates the methodologies underpinning the process. The SARB should assign ownership of this function, ensuring that staff become familiar with the model and run it on a regular basis, including to meet their commitment to recalibrate the buffers annually. The macroeconomic inputs to the CBST should be consistent with the internal macroeconomic projections used for policy making.

Table 1. Key Recommendations

Recommendations	Authority	Timeframe
1. Assign responsibility, identify dedicated staff, and build relevant modeling capacity, including at-risk methodologies for inflation and FX.	SARB	Near-term
2. Further develop satellite models (i.e., for CiC).	SARB	Medium-term
3. Determine and agree with the NT on risk appetite appropriate for the calibration of GFECRA and Contingency Reserve buffers.	SARB/NT	Near-term
4. Run the model at least annually using different macro scenarios to recalibrate the risk-based buffers.	SARB	Medium-term

Note: Near-term: < 12 months; Medium-term: 12 to 24 months.

Introduction

1. **SARB has unique financial arrangements.** It is privately owned, with fixed statutory capital, and there is no explicit recapitalization clause. The government has a direct claim on revaluation gains that accumulate in the Gold and Foreign Exchange Contingency Reserve Account (GFECRA) and is obligated to top up any losses that may accrue in that account, as it did in 2003. The FX risks of the SARB's holdings of foreign reserves are therefore borne by the National Treasury (NT). These reserves are sizable, representing 89 percent of assets (March 2024), while the GFECRA represents 40 percent of the balance sheet (liabilities plus equity).[2] The GFECRA consists of both realized and unrealized gains, is a liability to the NT, and distributions from it can only be made with the agreement of the SARB and the NT. Prior to the February 2024 budget, GFECRA was around ZAR 500 billion (7 percent of GDP).

2. **An MCM mission in December 2023 assessed the SARB's financial arrangements, with a focus on the GFECRA.** That mission emphasized that any GFECRA distributions should be grounded in the SARB's legal framework while underpinning its policy solvency. It demonstrated that any such distribution, if not financed by the sale of FX reserves (which could be ruled out, given that the actual levels were below those desired), would directly impact profitability since it would be funded by increased indebtedness to the banking system (i.e., higher level of excess reserves), upon which interest costs are incurred at the repo rate.

3. **The mission provided several recommendations, including:** (i) for the SARB and NT to agree on a high-level framework for GFECRA distributions; (ii) to establish risk-based targets for the GFECRA and the contingency reserve;[3] and (iii) to define an accounting policy for the segregation of realized and unrealized gains in the GFECRA. It did not, however, quantify the appropriate levels of buffer in either the GFECRA or contingency reserve, and it did not provide a methodology for the projection of the SARB balance sheet.

4. **This report extends the previous mission's work in the following areas by:** (i) quantitatively assessing an appropriate GFECRA buffer; (ii) assessing the adequacy of the contingency reserve; and (iii) helping to institutionalize a process to project the SARB balance sheet under different macroeconomic scenarios, and to recalculate the two buffers annually. To do this, a Central Bank Stress Test (CBST) model is applied to the SARB's balance sheet with both base and adverse scenarios, using at-risk methodology to illustrate outcomes based on identified levels of risk tolerance. It is emphasized that the SARB needs to determine its level of risk tolerance for the buffers to be calibrated. This requires an assessment of the trade-off between buffers, which could be seen as too large—a suboptimal allocation of public resources—or as too small—which could bring unwanted attention if one or both buffers were to be fully exhausted.

[2] These percentages are computed based on the balance sheet as of 2024Q1 in Table 2

[3] In this report, the term "contingency reserve" is used interchangeably with "equity buffer."

I. Developments Since the 2023 Mission

5. **The SARB implemented several of the 2023 mission's recommendations and developed a framework for GFECRA distributions in collaboration with NT.** It also set GFECRA (explicitly) and contingency reserves (implicitly, by virtue of the ZAR 100 billion transfer) buffers while publicly committing to an annual recalibration of both buffers. While most recommendations have been implemented, the SARB's approach to GFECRA differs from what was recommended, in that they did not differentiate between realized and unrealized gains, and only distributing the realized component. A complication with GFECRA arises given that it includes both realized and unrealized gains, therefore requiring retrospective analysis to identify the split between the two components—a task complicated by the extended period over which the gains have accrued. Instead, the SARB's approach aims to maintain a GFECRA buffer that, plausibly, would not be unwound by rand appreciations (point *iii* below). This is, in effect, a risk-based approach that is consistent with maintaining policy solvency—a core message from the December 2023 mission and something that is encapsulated in the principles agreed between the SARB and the NT (discussed below).

6. **The SARB and the NT announced the principles underpinning the new framework in the February 2024 budget:**[4]

    i. The SARB's policy solvency should not be undermined by any GFECRA distribution.

    ii. There should be no sales of FX to realize GFECRA gains if such reserves are below estimated adequacy levels.

    iii. There should be no distribution of unrealized GFECRA balances that could plausibly be unwound by future rand appreciations.

    iv. GFECRA distributions will be used to reduce government borrowing.

    v. Any GFECRA distributions should be governed by a framework that rules out ad hoc decisions.

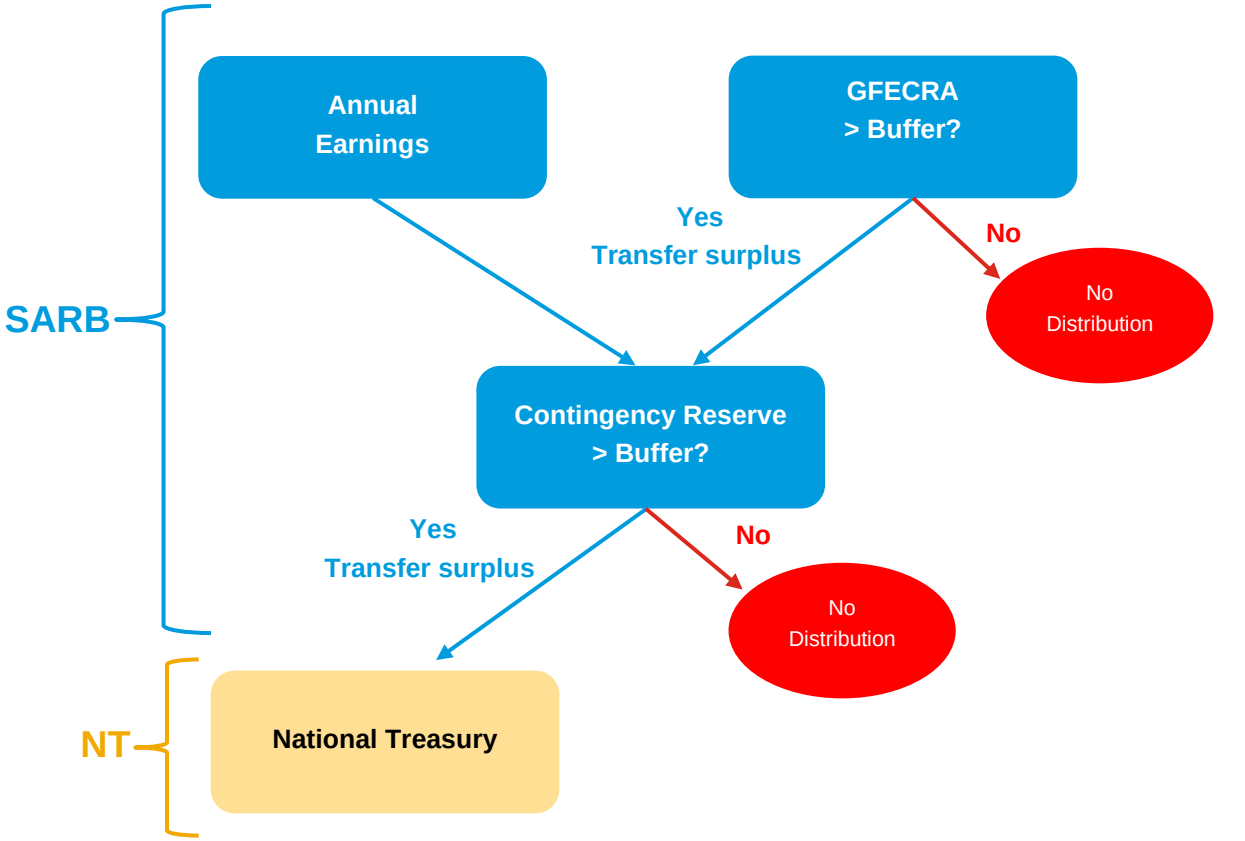
    vi. GFECRA settlement arrangements should be public to ensure transparency.

7. **The new framework for distributions involves a "waterfall" arrangement (Figure 1).** The GFECRA buffer is calibrated to be able to absorb large and plausible rand appreciations. Once this buffer is met, surplus balances are transferred to the contingency reserve, which is also set with a buffer. This buffer is calibrated to meet possible losses arising from SARB activities, including monetary operational costs (i.e., interest on excess reserves). Once this buffer is satisfied, surplus funds are then transferred to the NT. Based on a preliminary analysis, the SARB and the NT agreed that the GFECRA buffer for the first year would be set at ZAR 250 billion, allowing for distribution of ZAR 250 billion. A transfer to the NT of ZAR 150 billion over three years was also agreed upon, with ZAR 250 billion paid by the SARB to the NT and ZAR 100 billion transferred to the SARB from the National Treasury. ZAR 100 billion was transferred to the Contingency Reserve,[5] although no explicit target for this buffer has been set.

[4] See NT/SARB media release, "Gold and Foreign Exchange Contingency Reserve Account (GFECRA) Frequently Asked Questions," February 23, 2024.

[5] This transfer was authorized by amendment to the Gold and Foreign Exchange Contingency Reserve Account Defrayal Act, 2003, which provides for direct charges against the National Revenue Fund for the requirements of the South African Reserve Bank.

Figure 1. Annual Distribution Framework



Source: IMF staff.

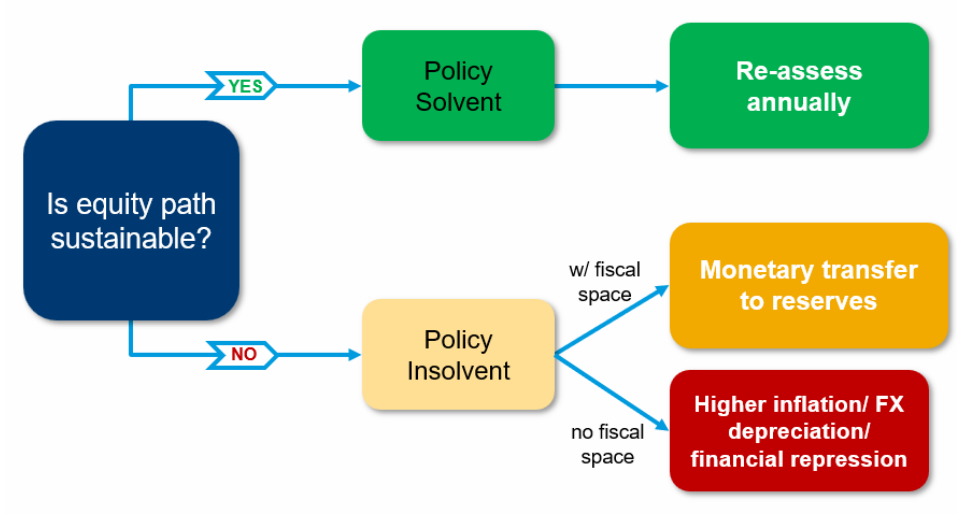
II. Central Bank Stress Test Framework

A. Policy Solvency

8. **A central bank is assessed to be policy solvent if it is on a sustainable path, which is a condition that requires positive realized earnings, on average, over time.** The absolute level of equity is somewhat less important because there are a few examples of central banks operating effectively with negative equity. These are, however, specific cases with strong institutional frameworks, central banks with underlying (i.e., realized) profitability, and where governments had strong fiscal positions.

9. **If the central bank is not policy solvent, there are two possible outcomes (Figure 2).** If the government has the fiscal space to make a monetary transfer to the central bank's reserves, then it should do so. Alternatively, if the government does not have the fiscal space, then the outcomes are limited to either financial repression (e.g., capital controls, negative real interest rates) or a loss of control over financial conditions (e.g., the exchange rate), and consequently over inflation.

Figure 2. Assessing Policy Solvency



Source: IMF staff.

B. Overview

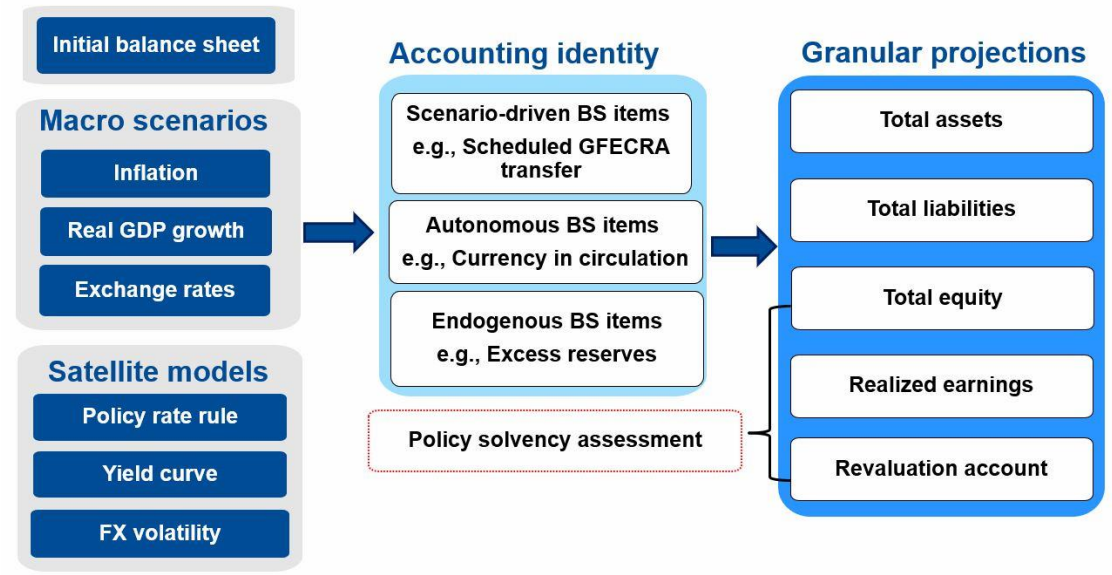
10. **The central bank stress test model generates the projections of the balance sheet items from the initial balance sheet, using scenarios of macroeconomic variables (Figure 3).** Projections of real GDP, inflation, and the exchange rate are centered on area department assumptions, while the policy rate is based on a Taylor rule. The model incorporates accounting rules and equations derived from macroeconomics and finance. All material items on the balance sheet are modeled, deriving a path for total equity that is split between retained earnings and the revaluation account. The items can be classified into three groups:

    i. Exogenous or policy-determined items. These include net purchases of the foreign currency and investments in domestic government bonds. These items are not estimated, but derived from policy decisions (e.g., regarding the target level of net international reserves).

    ii. Autonomous items determined by macroeconomic variables. These do not explicitly depend on other items on the balance sheet (i.e., CiC). They are items outside the direct and immediate control of the central bank and thus are not directly related to monetary policy operations but can influence the amount of liquidity in the banking system. Therefore, they need to be forecasted using satellite models.

    iii. Endogenously determined items. Excess reserves are the key endogenous item because it is determined by changes in CiC and required reserves on the liability side, and net purchases of the USD and investment in the government bonds on the asset side. Excess reserves also impact interest expenses and operating costs.

Figure 3. Modeling the Central Bank Balance Sheet



Source: IMF staff.
Note: BS means Balance Sheet.

C. Core Equations

11. **Three core equations capture the dynamics of the central bank balance sheet: equity, a clearing item (e.g., excess reserves), and the foreign reserves.** These equations conform to accounting identities and hold independently of the macroeconomic assumptions.

12. **Total equity can be obtained with two methods.** The model should have a cross-check for internal consistency by looking at the values of total equity from these two methods:

$$(1)\ \text{Equity}_t = \text{Total Assets}_t - \text{Total Liabilities}_t$$

$$(2)\ \text{Equity}_t = \text{Equity}_{t-1} + Realized\ earnings_t + Reval\ Account_t$$

13. **Depending upon the operational framework, either excess reserves or open market operations is a clearing item;** this item captures all changes in the assets and liabilities that impact realized earnings. Revaluation gains or losses are not captured because they go directly against the revaluation account. The changes in the balance sheet (that impact realized earnings) are reflected in changes in the excess reserves. This is given by: (i) the net USD purchases for accumulating the foreign reserves, ΔFXInt_t; (ii) the net increases in the government bond investments, $\Delta\text{GovBond}_t$; (iii) the transfer from GFECRA to the NT and the dividends to the NT, which are added to the excess reserves, ΔNT; (iv) changes in CiC, $\Delta\ \text{CiC}_t$; (v) changes in the required reserves, $\Delta\ \text{RR}_t$; and (vi) interest expenses, IRExpense_t, and operating costs, Op_t.[6] In the case of SARB, the clearing item is the excess reserves, ResEx, which pays interest at the policy rate.

[6] Operating costs increase in proportion to the nominal GDP. As the operating costs are relatively small compared with other expenses, our main results remain qualitatively the same, even if we use different specifications for operating costs.

$$\mathrm{ResEx_t} = \mathrm{ResEx_{t-1}} + [\,\Delta\mathrm{FXInt_t} + \Delta\mathrm{GovBond_t} + \Delta\mathrm{NT} - \Delta\,\mathrm{CiC_t} - \Delta\,\mathrm{ResReq_t} + \mathrm{IRExpense_t} + \mathrm{Op_t}]$$

14. **The dynamics of gross foreign reserves are driven by three terms: (i) foreign currency interest income, (ii) FX revaluation gains or losses, and (iii) net USD purchases.** The equation below is shown in the simplest case. For example, when there is net inflow from other sources, there is an additional term to account for the equivalent change in the gross foreign reserves. The assumption is that there is no such inflow, but the model is flexible to allow other drivers. In the equation below, $\mathrm{FXRes_t}$ is the foreign reserves at time t. $\mathrm{FIRIncome_t}$ is the foreign interest income. $\mathrm{Revaluation_t^{FXRes}}$ is the revaluation gain due to the change in exchange rates. $\Delta\mathrm{FXInt_t}$ is the net amount of the purchase of foreign currencies, which is primarily USD.

$$\mathrm{FXRes_t} = \mathrm{FXRes_{t-1}} + \mathrm{FIRIncome_t} + \mathrm{Revaluation_t^{FXRes}} + \Delta\mathrm{FXInt_t}$$

D. Satellite Models

15. **Satellite models are developed for three items that have material impacts on equity dynamics.** Unlike the core model equations described above, assumptions are based on specific macroeconomic circumstances. What follows are parsimonious but still sufficiently flexible models for the key balance sheet items. These satellite models are replaceable with simpler models or more sophisticated models, depending on the purposes of CBST model users.

16. **CiC is assumed to linearly increase with nominal GDP and depends on the policy rate.** The economic intuition is that a larger economy requires more CiC for transactional purposes, and households hold less cash if interest rates are higher (i.e., opportunity cost). The policy rate serves as the proxy for the deposit rate. The interest-sensitivity parameter is calibrated using historical data. We employ this specification to keep the satellite model simple while capturing the key effect of the interest rate. For example, structural breaks in the relationship between CiC and the nominal GDP can be captured by allowing the parameter η_{CiC} to change over time. However, it requires estimating additional parameters and makes the model more complicated.

$$\mathrm{CiC_t} = \eta_{\mathrm{CiC}}\,\mathrm{NGDP_t} \cdot \left(\frac{\mathrm{i_0}}{\mathrm{i_t}}\right)^{\gamma_{\mathrm{CiC}}}$$

17. **Required reserves in ZAR are assumed to increase with nominal GDP.** This is because required reserves increase as the total bank deposits grow (given a constant ratio on the reserve requirement), and total deposits increase with nominal GDP. Regarding the renumeration, there are two options for model users. The first is zero interest rate on the required reserves. The second option is interest paid on required reserves is calculated at the policy rate. The first option is used because it is consistent with the current institutional setting.

$$\mathrm{ResReq_t} = \eta_{\mathrm{Req}}\,\mathrm{NGDP_t} + \mathrm{i_{req,t-1}}\mathrm{ResReq_{t-1}}$$

18. **A simplified Taylor rule is used to generate a policy rate path given the projections of macroeconomic variables.** Specifically, we assume that the nominal policy rate mean-reverts to its target level $\bar{\imath}_t$.

$$i_t = \rho_{\mathrm{i}} i_{t-1} + (1-\rho_{\mathrm{i}})\bar{\imath}_t + \sigma_i \epsilon_{i,t},$$

Where ρ_{i} is the autoregressive coefficient and σ_i is the volatility. $\epsilon_{i,t}$ is sampled from the standard normal distribution. The target policy rate level $\bar{\imath}_t$, is determined by the following: inflation π_t, the

target inflation level $\bar{\pi}$, real GDP growth y_t, the mean-reverting level of the real GDP growth $\bar{y}$, and constant real rate r^*.[7]

$$\bar{\imath}_t = \bar{\pi} + r^* + \gamma_\pi(\pi_t - \bar{\pi}) + \gamma_y(y_t - \bar{y}).$$

E. Application to the SARB

19. **The CBST framework is adapted to the SARB's institutional arrangements.** The most notable, unique feature is GFECRA has the FX revaluation accounts separated from equity. GFECRA incorporates revaluations arising from changes in FX spot prices and gold prices.

20. **Transfers from GFECRA to equity are modeled, as well as transfers of equity to the NT.** Specifically, the excess of the equity buffer is added to the excess reserves. This reflects the fact that the transfer goes to the deposits at commercial banks, not the deposits from the government at SARB.

21. **Two different specifications for the GFECRA buffer are as follows.** The first specification is prespecified deterministic function of time. The second is Value-at-Risk (VaR) for FX reserves. VaR is the threshold that determines the worst loss that the central bank wants to consider.[8] The buffer for each currency is computed given the current exposure and sum of the buffers across currencies, to obtain the total GFECRA buffer.[9] Mathematically, the GFECRA buffer is computed as:

$$\text{GFECRA Buffer}_{\text{t}} = \sum_{\text{i=1}}^{\text{N}} E_i \cdot \left(1 - \exp((\mu_i - 0.5\sigma_i^2)\tau + \sqrt{\tau}\sigma_i VaR(\alpha))\right)$$

Where μ_i is the average log return of the exchange rate (e.g., ZAR/USD); σ_i is the volatility of the log return of the exchange rates; τ is the time horizon, which determines the number of years for which the same exposure is assumed to be held; and α is the percentile of VaR, which determines the probability of the loss that exceeds the threshold. $VaR(\alpha)$ is assumed to be a percentile of the standard normal distribution to keep the model simple.

22. **The adequacy of the equity buffer is assessed by computing the number of years it takes to deplete the equity.** Specifically, we apply at-Risk scenarios and then compute the time at which the equity reaches zero.[10] The initial amount of the equity is 42 billion ZAR to which is added 100 billion ZAR as announced in the 2024 budget—this is added to the contingency reserve, which is a part of the equity. The task, then, is to calculate how long it takes for the equity (ZAR 142 billion) to be fully depleted given projections of negative realized earnings.

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[7] The constant real rate can be interpreted as an equilibrium real rate or natural rate in a broad sense. The parameters in the Taylor rule are made consistent with SARB's internal macro forecasting model.

[8] In the literature of mathematical finance, it has been argued that expected shortfall is better than VaR because it is more conservative and satisfies the mathematical property of subadditivity. We consider VaR a reasonable modeling assumption in the context of SARB for two reasons. First, one can always adjust the percentile of VaR so that VaR is sufficiently conservative, compared with expected shortfall. Second, expected shortfall is difficult to estimate and statistically test because it is not directly observable.

[9] In this specification, the dependence between the exchange rates is not modeled for simplicity and conservativeness. In other words, the specification assumes that the correlations between a pair of two currencies are equal to one because the same percentile across currencies is assigned.

[10] When the equity path does not reach zero during the time horizon of the projection (i.e., five years), the equity path is extrapolated linearly to compute the time when the equity reaches zero.

23. **The approach for calibrating the equity buffer differs from that used for the GFECRA due to computational challenges.** The initial level of the equity needs to be solved for, such that it remains above zero with a certain probability in the future. Mathematically, it is formulated as follows. Consider the future equity as a function of the initial balance sheet, BS_t, the macro scenarios, Macro_t, the preestimated parameters, θ, and the equity buffer, $\mathrm{EquityBuffer}_t$, under the CBST framework.

$$\mathrm{E}_{t+\tau}^{\alpha} = \mathrm{f}_{\alpha}(\mathrm{EquityBuffer}_t, \mathrm{BS}_t, \mathrm{Macro}_t, \mathrm{Z}_t, \theta, \tau).$$

To establish this function, a Monte Carlo simulation is run with stochastic shocks, Z_t, for the central projection of macro variables. Given the VaR percentile α, and the time horizon τ, the target equity buffer is obtained by solving the following equation.

$$0 = \mathrm{f}_{\alpha}(\mathrm{EquityBuffer}_t^*, \mathrm{BS}_t, \mathrm{Macro}_t, \mathrm{Z}_t, \theta, \tau).$$

24. **As equity is a highly nonlinear function of the macro scenarios and the initial balance sheet, a numerical method is used to obtain the amount of the target equity buffer.** Instead, the time horizon τ^* is solved, mathematically represented as:

$$\mathrm{E}_{t+\tau^*}^{\alpha} = \mathrm{f}_{\alpha}(\mathrm{EquityBuffer}_t, \mathrm{BS}_t, \mathrm{Macro}_t^{\mathrm{VaR}}, \theta, \tau^*) = 0,$$

Where $\mathrm{Macro}_t^{\mathrm{VaR}}$ is the prespecified at-Risk scenario.[11]

25. **Two further different specifications for the equity buffer are also implemented for potential future use.** The first specification is a prespecified deterministic function of the time. The CBST model user can specify how the equity buffer evolves over time as a deterministic path. The second specification is a function of the policy rate and the excess reserves. Mathematically, the equity buffer is written as follows.

$$\mathrm{EquityBuffer}_t = \gamma \cdot i_t \cdot ExcessReserve_t\ ,$$

Where the parameter γ is how many years it takes to deplete the buffer. A higher γ leads to more conservative buffer.

26. **The SARB balance sheet is strong and mainly consists of foreign reserves with small interest-bearing liabilities** (**Table 2**). On the asset side, all assets are income generating; 89 percent are foreign assets, and the remaining are domestic assets. On the liability side, the excess reserves, which is the key interest-bearing item, are 6 percent of the total liabilities. An aggregation of the balance sheet (Table 3) helps to understand the key driving factors of the balance sheet dynamics.

[11] Since the at-Risk scenario is prepared outside of the CBST model, the Monte-Carlo simulation is not conducted and thus there is no stochastic variable in the function.

Table 2. SARB Balance Sheet as of 2024Q1

Assets	ZAR (thousands)
Gold and foreign exchange reserves	1,179,190,360
Cash and Cash equivalent	61,743,000
Investments	23,637,000
Loan and advances	7,080,000
South African government bonds	32,007,000
All other assets	22,735,000
Total assets	1,326,227,000

Liabilities	ZAR (thousands)
Currency-in-circulation	169,504,000
Deposit accounts: Required reserves	151,508,000
Deposit accounts: Excess reserves	80,095,000
Deposit accounts (from others)	128,545,000
Deposit accounts (from gov't)	114,124,000
Foreign deposits (from gov't): USD	81,625,263
Foreign deposits (from gov't): EUR	9,675,810
Foreign deposits (from gov't): CNY	2,112,877
Foreign deposits (from gov't): Other ccys	6,802,930
GFECRA	531,988,550
All other liabilities	8,259,570
Total equity	41,986,000
Total liabilities plus equity	1,326,227,000

Source: SARB and IMF staff.

Table 3. Aggregated Balance Sheet

Asset	ZAR (thousand)
Gross foreign asset	1,179,190,360
Net claim on others	22,569,640

Liabilities	ZAR (thousand)
Currency in circulation	169,504,000
Net claim to government	158,696,880
GFECRA	531,988,550
Net claim to banks	162,780,000
Net claim to others	136,804,570
Total equity	41,986,000

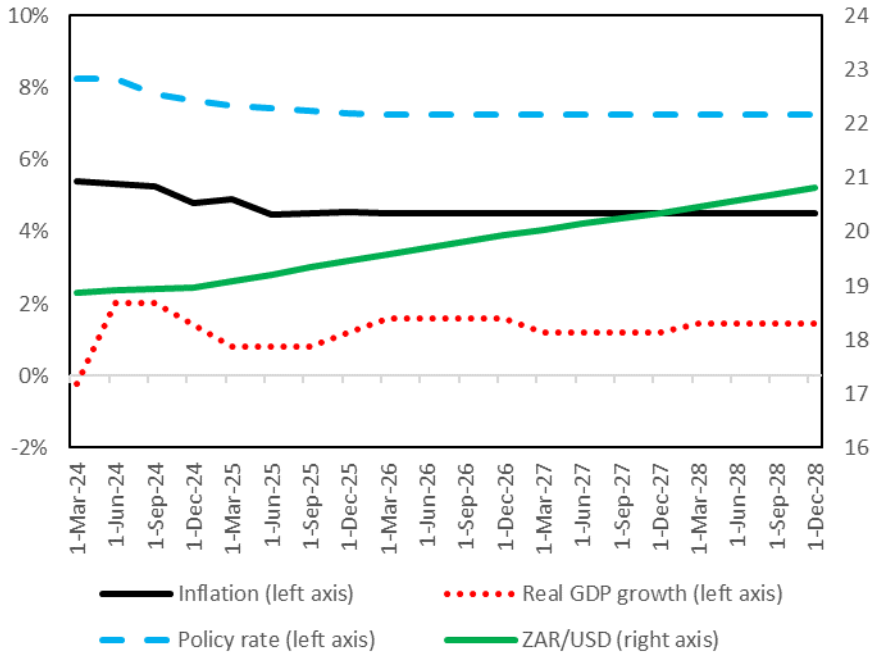
Source: SARB and IMF staff.

III. Results

A. Base Scenario

27. **In the base scenario, the South African economy is assumed to grow slowly but steadily (Figure 4).** Real GDP growth is expected to grow to 1 percent in 2024. Given dissipating election uncertainty, economic activity is expected to accelerate in the second half of the year, driven by a rebound in investment and private consumption. Growth is projected to stabilize at 1.4 percent in the medium run, assuming structural bottlenecks are being eased but will not be eliminated, absent an acceleration of structural reforms.

Figure 4. Macro Assumptions in Base Scenario

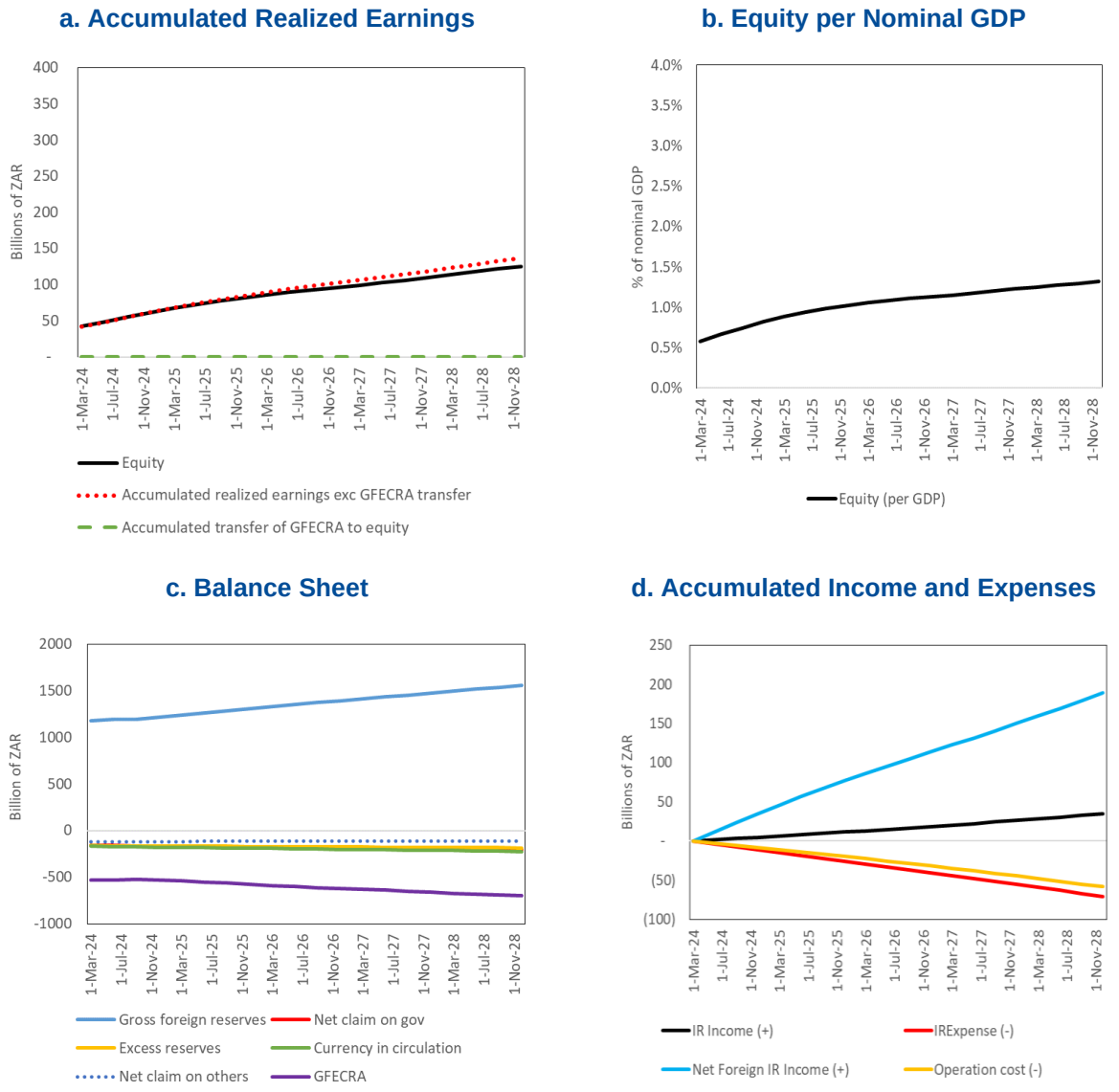


Source: IMF staff.

28. **In the absence of any distributions of GFECRA, equity reaches 1.3 percent of GDP by 2028Q4 (Figure 5).** Four observations are made: (i) equity increases over time in both absolute terms and per nominal GDP; (ii) the accumulated FX revaluation gain included in the equity does not change over time because almost all FX revaluation gains are included in GFECRA;[12] (iii) both net foreign reserves (NFR) and GFECRA increase over time because of depreciation of ZAR against USD in the base scenario; and (iv) foreign interest income is larger than domestic interest expenses on the excess reserves and operational costs, and thus accumulated realized earnings increase over time.

[12] The FX revaluation gain from the foreign deposits of the government on the liability side is not included in GFECRA. The size of this foreign deposit is small (8 percent of the total liabilities).

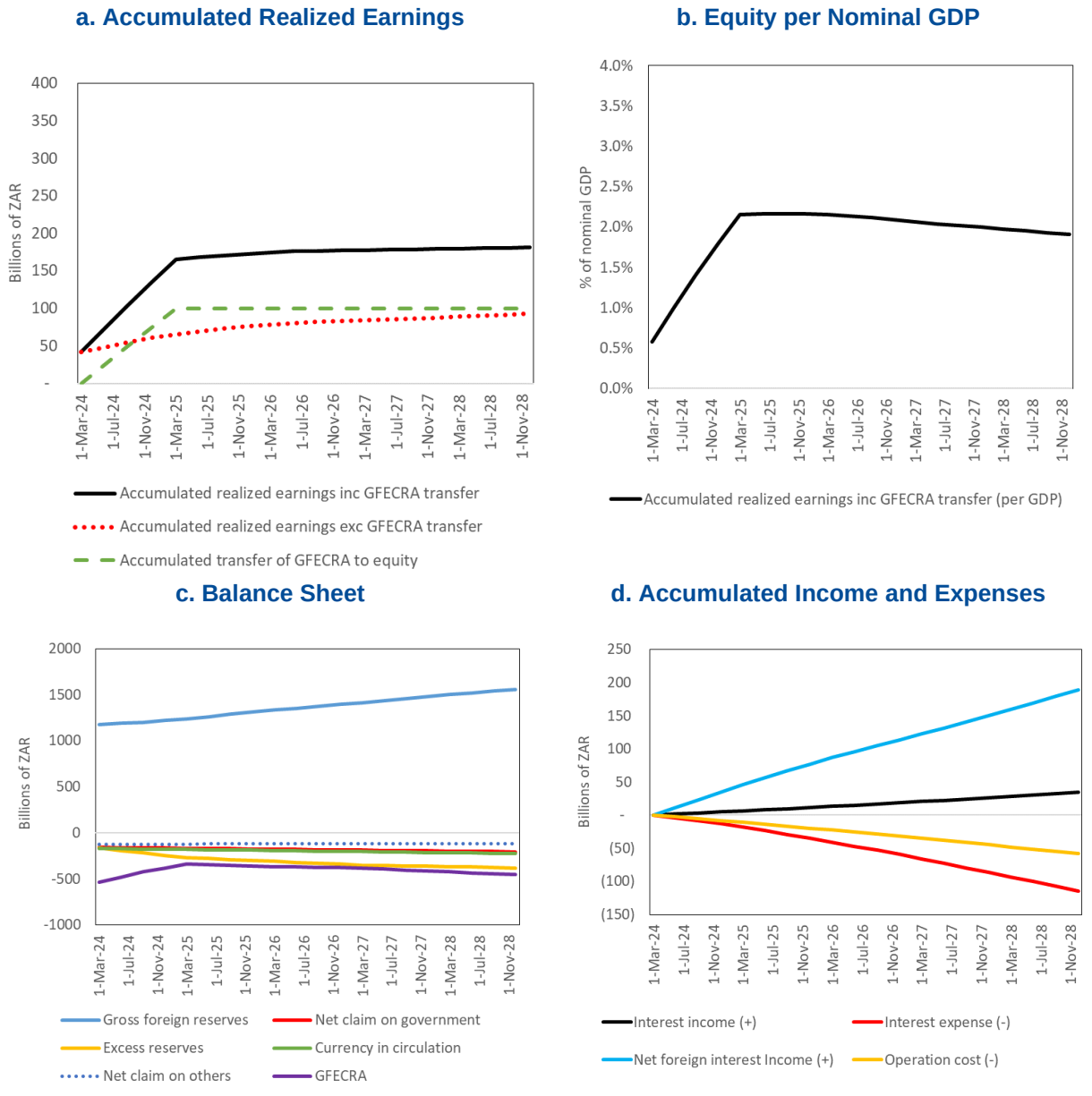
Figure 5. Results in Absence of Distribution of GFECRA under Base Scenario



Source: IMF staff.

29. **The SARB remains policy solvent with the scheduled transfers of GFECRA to equity and the excess reserves, as announced in the February 2024 budget statement (Figure 6).** Equity will reach 1.9 percent by 2028Q4, under the base scenario. Several observations are noted. Equity increases rapidly by around 100 billion during the 2023–2024 period due to the transfer from GFECRA to the contingency reserve. It continues to increase after 2024, as realized earnings increase, indicating policy solvency under the base scenario. Realized earnings increase because the foreign and domestic interest income exceeds the domestic interest expenses and operating costs. GFECRA declines by ZAR 250 billion because of its distribution to the contingency reserve and the excess reserves, and starts to increase with continued ZAR depreciation, which results in FX revaluation gains over time.

Figure 6. Results with Scheduled Distribution of GFECRA under Base Scenario



Source: IMF staff.

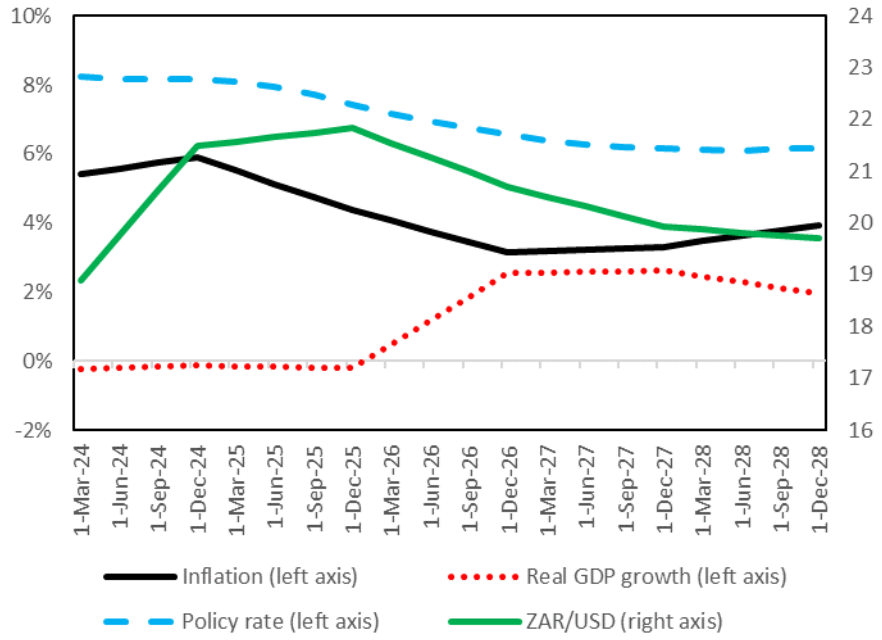
B. Adverse Scenario

30. **The adverse scenario is characterized by higher inflation and a mild depreciation shock to ZAR, arising from slowdown in the global economy (Figure 7):** A slowdown in China's economy, accompanied by disinflation, leads to lower global demand but also improved terms of trade, with a mixed effect on Emerging Market exports. An escalation of the conflict in the Middle East results in a surge in oil prices and shipping costs, amounting to a negative global supply shock. Tighter monetary policy in the United States leads to higher sovereign and corporate

premia and currency depreciation in emerging markets. As a result of these combined shocks, lower private consumption and investment would lead to a decline in South Africa's growth.

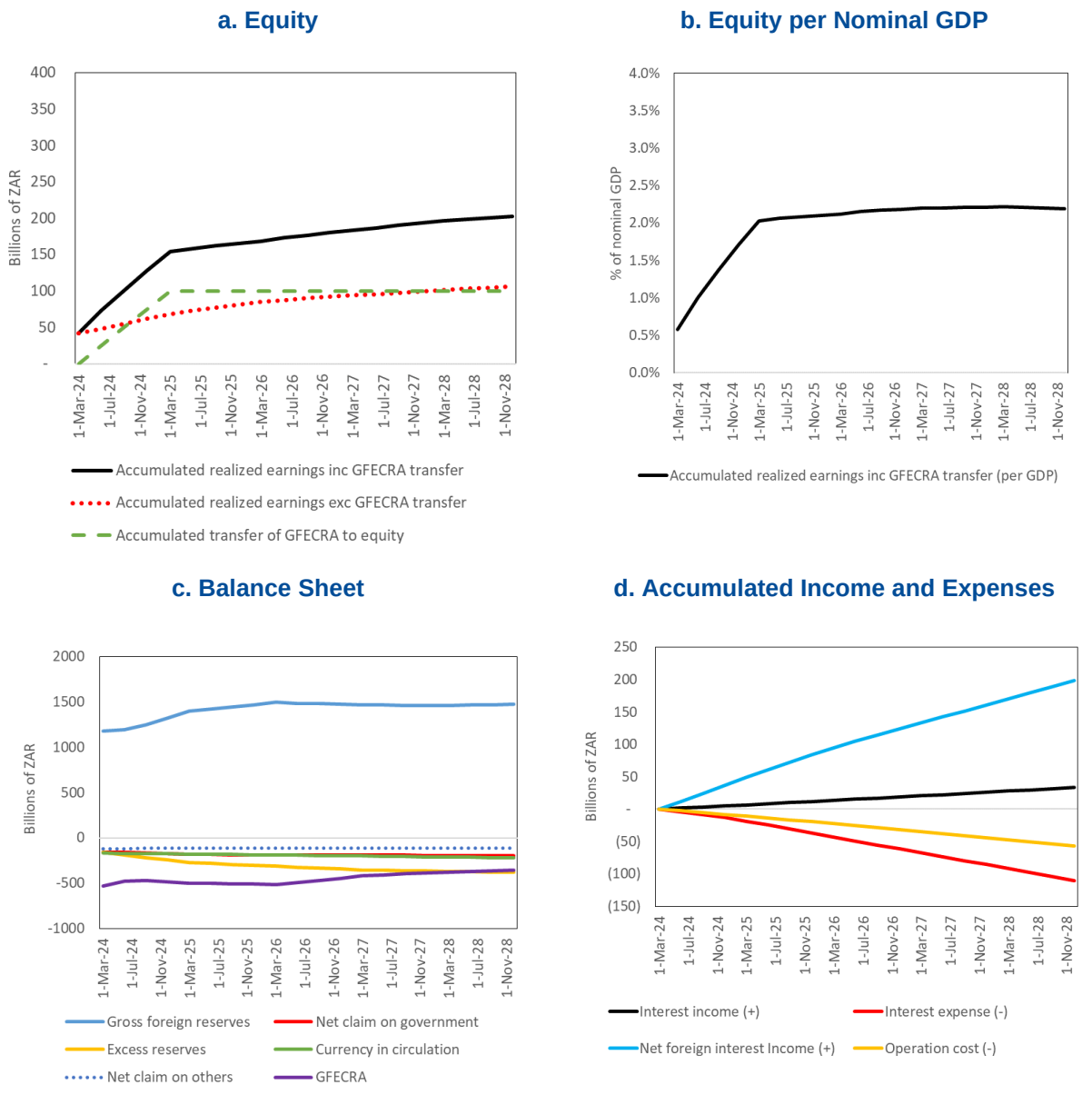
31. **Equity increases more rapidly under the adverse scenario than under the base scenario primarily because of the larger foreign interest income (Figure 8).** There are two opposing effects. The positive effect is that a larger depreciation of ZAR leads to an increase in the foreign interest income. The negative effect is that higher inflation leads to larger interest expenses due to a higher policy rate. The positive effect outweighs the negative effect, with equity reaching 2.2 percent by 2028Q4. The result shows that the adverse scenario, from the view of the macroeconomy, is not adverse for the SARB's equity, and thus indicates that the scenarios need to be designed specifically for stress testing the SARB's balance sheet.

Figure 7. Macro Assumptions in Adverse Scenario



Source: IMF staff.

Figure 8. Results under Adverse Scenario



Source: IMF staff.

C. At-Risk Scenarios

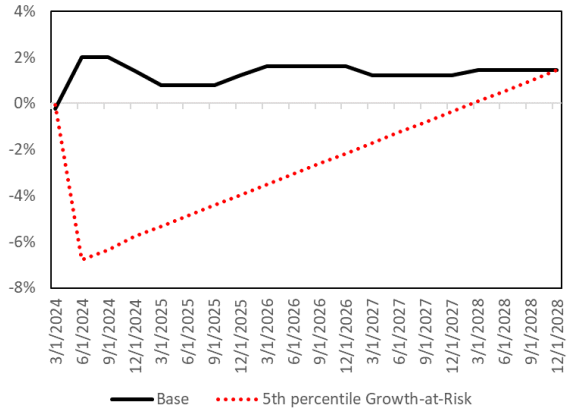
32. **Three at-Risk scenarios are employed to stress test the balance sheet:** Inflation-at-Risk, Growth-at-Risk, and FX-at-Risk. The main area of interest is FX-at-Risk because the primary risk in the balance sheet is FX risk.

33. **Equity increases faster under the Growth-at-Risk scenario than under the base scenario because the interest expenses are smaller due to a lower policy rate (Figure 9).** The Real GDP growth declines to –7 percent in 2024, and then recovers to the same level as the base case. The negative real GDP growth shock leads to a lower policy rate and thus smaller interest

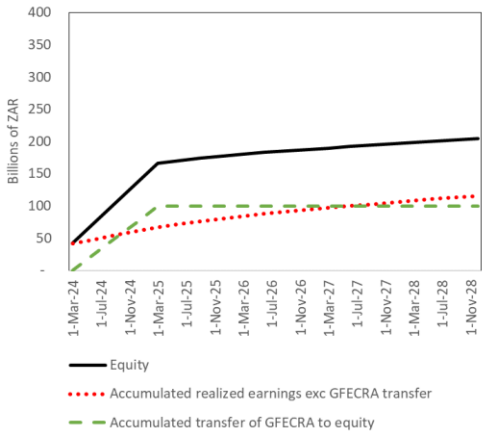
expenses under Growth-at-Risk. This helps equity to increase faster. The equity will reach 2.6 percent in 2028Q4.

Figure 9. Growth-at-Risk (5th percentile)

a. Real GDP Growth



b. Equity

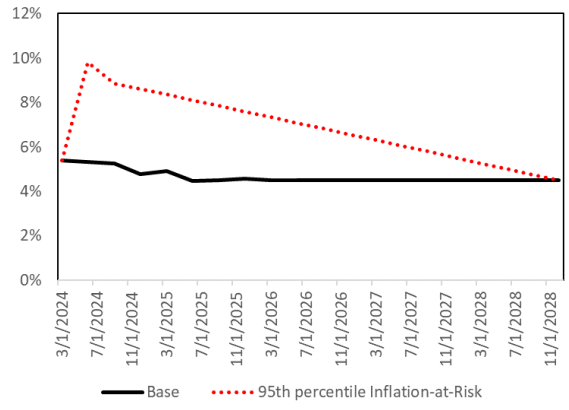


Source: IMF staff.

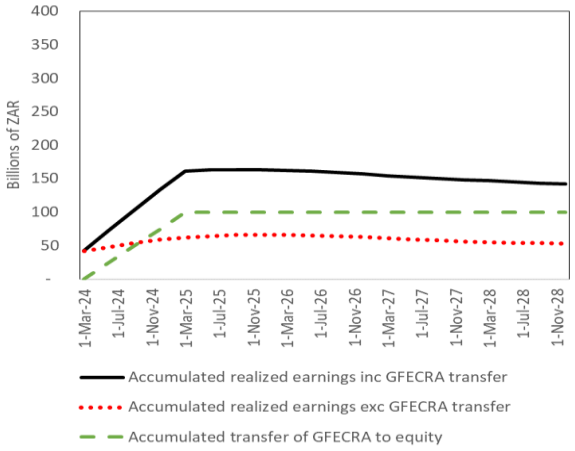
34. **Equity decreases slowly over time, under the 95th-percentile Inflation-at-Risk scenario, because the interest expenses are larger due to a higher policy rate (Figure 10).** Inflation rises to 10 percent in 2024 and then declines to the same level as the base case. The adverse inflation shock leads to a higher policy rate and thus larger interest expenses under Inflation-at-Risk. The equity will reach 1.4 percent in 2028Q4.

Figure 10. Inflation-at-Risk (5th percentile)

a. Inflation



b. Equity

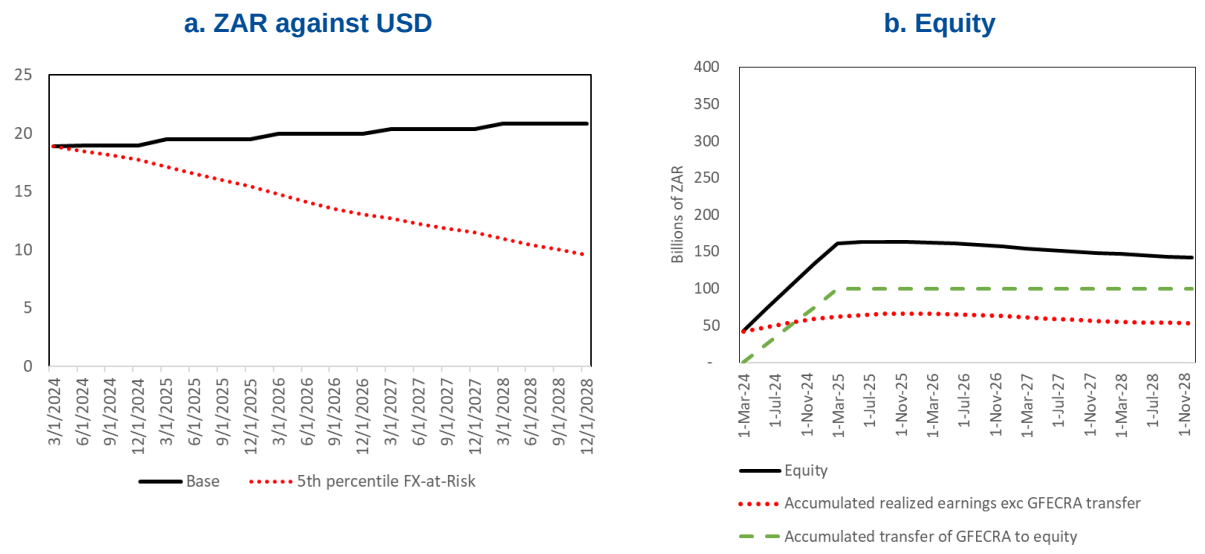


Source: IMF staff.

35. **Equity decreases slowly over time, under the 5th-percentile FX-at-Risk scenario, because the foreign interest income is smaller due to the appreciation of ZAR against USD (Figure 11).** ZAR continues to appreciate against USD by 49 percent over the five years. We consider the

magnitude of the appreciation not unrealistic for the purpose of the stress testing because, for example, ZAR was appreciated against USD by 52 percent during the 2001/12–2004/11 period. The appreciation of ZAR makes the foreign interest income smaller under FX-at-Risk. Consequently, the accumulated realized earnings are smaller under this scenario than under the base scenario. The equity will reach 1.4 percent in 2028Q4.

Figure 11. FX-at-Risk



Source: IMF staff.
Note: The forecasts of FX-at-Risk use moving averages in order to smooth the seasonal patterns for long-term projections.

D. Quantifying the GFECRA Buffer

36. **The VaR-based approach is used to quantify the GFECRA buffer against FX appreciation risk.** To do so, the mean and the volatility of the log return of the exchange rates are estimated. The VaR formula is then applied for each currency and the VaR-based unexpected revaluation losses are aggregated across all currencies.

37. **A smaller percentile and longer time horizon increases GFECRA buffer (Table 4).** The corresponding log return for the combination of selected percentile and time horizon is shown for reference. It is noteworthy that the magnitude of these returns is below the historical worst returns from January 2000 to July 2024. For example, the three-year log return from 2001/12/31 to 2004/11/30 is −72.5 percent.

38. **The SARB's current calibration of the GFECRA buffer (ZAR 250 billion) is close to the 5th-percentile with 2-year time horizon (243 billion ZAR).** This choice would seem to be both conservative and appropriate. The adequacy of the buffer should reflect several factors: (i) the size of the exposure for each currency; (ii) the exchange rate uncertainty, measured by the statistical moments such as the volatility of the exchange rates; and (iii) the risk tolerance parameters, such as the percentile and the time horizon of VaR. These risk tolerance parameters are based on SARB's policy decision. It is recommended that the SARB employ a quantitative approach as described above, in order to capture these factors in a systematic way so that annual recalibrations of the buffer, as publicly committed to, can be systemically explained.

Table 4. GFECRA Buffer under VaR-based Approach

Amount of GFECRA Buffer

ZAR bn	Time horizon		
percentile	1 year	2 year	3 year
1.0%	289	363	406
2.5%	241	300	333
5.0%	198	243	264
7.5%	169	203	217
10.0%	146	171	178

Corresponding Change in ZAR/USD (log return)

Return	Time horizon		
percentile	1 year	2 year	3 year
1.0%	-29.0%	-38.5%	-44.8%
2.5%	-23.8%	-31.1%	-35.7%
5.0%	-19.3%	-24.7%	-27.9%
7.5%	-16.3%	-20.5%	-22.8%
10.0%	-14.0%	-17.3%	-18.8%

Source: IMF staff.

E. Assessing the Adequacy of the Equity Buffer

39. **The adequacy of the current equity buffer (142 billion ZAR) is assessed by using the combination of the FX-at-Risk and Inflation-at-Risk scenarios.** The adequacy of the equity buffer is not directly computed due to the computational difficulty noted earlier. Rather, the number of years it takes to deplete equity is calculated. Note that the equity dynamics depend on the GFECRA, because the surplus over that buffer is transferred to the contingency reserve, which is part of the equity. To facilitate the interpretation of the results, we turn off the additional distribution of the GFECRA to the equity buffer beyond the scheduled transfer. For this purpose, the VaR-based GFECRA buffer is assumed at 1-percentile over a two-year horizon. This assumption is made for illustrative purposes: It is highly conservative and ensures that there is no additional distribution of GFECRA during the simulation time horizon, which simplifies the analysis on the equity buffer.[13]

40. **A lower percentile of FX-at-Risk and higher percentile of Inflation-at-Risk leads to fewer years to deplete equity (Table 5).** This means more appreciation of ZAR and lower foreign interest income. Higher percentile Inflation-at-Risk leads to higher inflation and thus increases the domestic interest expenses on the excess reserves. Therefore, the combination of these two scenarios depletes equity more quickly.

Table 5. Equity Buffer

Time Horizon to Deplete Equity

Inflation-at-risk (percentiles)	FX-at-Risk (percentiles)		
	1	5	10
99	13.2	16.1	18.3
95	14.9	18.8	21.9
90	15.9	20.5	24.4

Source: IMF staff.

[13] If the GFECRA buffer is set sufficiently low, then an additional distribution of GFECRA to equity occurs during the simulation time horizon. Consequently, the time horizon to deplete equity in Table 5 would be longer. In this sense, the result is a conservative estimate.

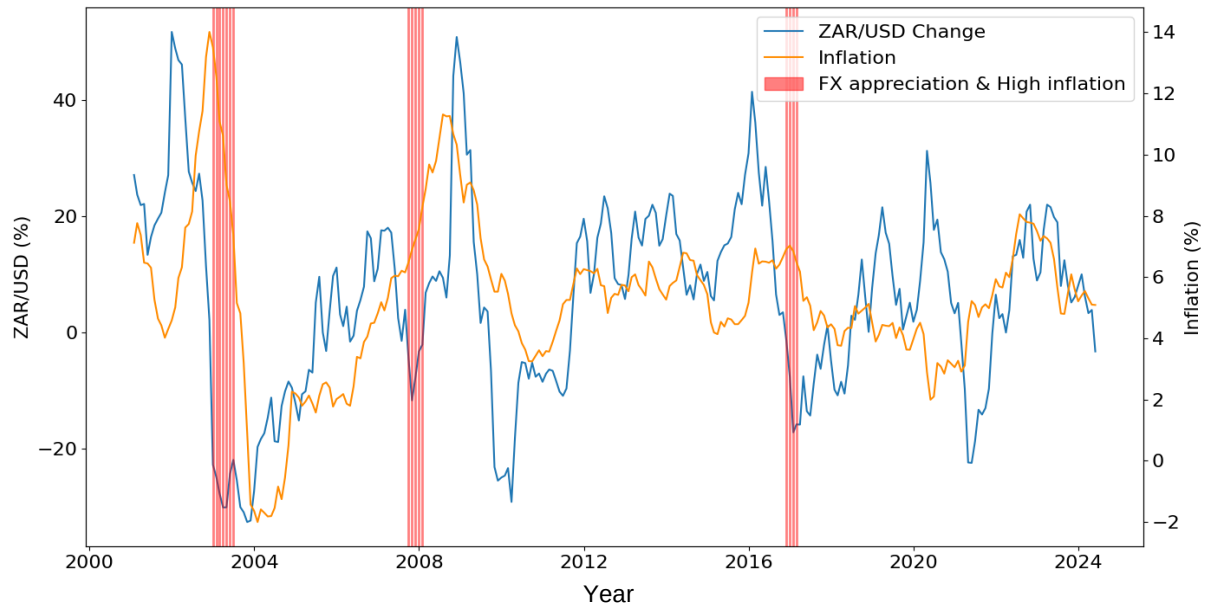
41. **Equity (currently ZAR 142 billion) will remain positive over the next 13 years under the severest combination of FX-at-Risk and Inflation-at-Risk.** The buffer could increase if the assumptions change. For example, it is assumed that foreign reserves remain constant in USD terms, but if SARB purchases USD over the next five years, the cost of financing the USD purchase adds to the excess reserves and thus leads to large interest expenses. Similarly, the risks of losses arising from the SARB's lender of last resort function are not captured, but the buffer could be higher if such risks were incorporated. Other factors such as changes in CiC dynamics would also affect buffer estimates.

42. **While the rand has been in a long-term downward trend, plausible scenarios exist for a currency appreciation, including with higher inflation and interest rates.** While expectations are for further rand depreciation, over time, there have been past episodes when the currency appreciated amid conditions of elevated inflation (Box 1). Such episodes are the worst combination for the SARB balance sheet, since they result in both lower foreign currency interest income, and higher domestic interest expenses. Furthermore, in recognizing that the USD is at or close to cyclical highs, a mean reversion coupled with some unwinding of the increase in South Africa's country risk premium since 2014, could result in a rand appreciation nearing 30 percent (Box 2).

Box 1. Combined Shocks of ZAR Appreciation and High Inflation

Historical trends indicate that South Africa has experienced simultaneous ZAR appreciation and high inflation. An analysis spanning from 2000 to 2024, presented in the figure below, pinpoints three specific periods, highlighted in red, during which the ZAR appreciated while inflation exceeded 75 percent of recorded instances.

Historical Trends of the Exchange Rate and Inflation



Source: SARB, IMF staff calculations.

Box 2. Possible Scenario of a Significant ZAR Appreciation

A scenario with FX appreciation is estimated to inform the impact on the U.S. dollar index and sovereign credit default swap (CDS) spread reductions to the exchange rate (USD in 1 ZAR). The equation of this model is as follows:

$$FX_t = \alpha_0 + \alpha_1 I_t + \alpha_2 CDS_t$$

Estimated Results of USD/ZAR Equation

	Estimate	Standard Error	t-value	Pr(>|t|)
α_0	0.0013	0.015	0.083	0.934
α_1	***-1.1936	0.054	-22.040	0.000
α_2	*** -0.1157	0.006	-19.301	0.000
Adjusted R-Square	0.385			
Durbin-Watson Test Statistic = 2.286				

Source: IMF staff calculations. *** indicates statistical significance at the 99.9-percentile level.

Where FX_t is the exchange rate (USD in 1 ZAR), I_t is the nominal broad U.S. dollar index, and CDS_t is the 5-year sovereign CDS spread. Daily data were used from 2014 to 2024. The result of the estimated equation is shown in the table above. Diagnostic tests showed there was no serial autocorrelation in the residuals.

The South African rand (ZAR) is expected to appreciate by approximately 29.73 percent, assuming the specified decreases in the U.S. dollar index and sovereign CDS from their current levels to historical levels at the beginning of 2014. Specifically, a 24 percent decrease in the U.S. dollar index from the current level to historical level seen in 2014 contributes to a 28.17 percent appreciation of ZAR. Simultaneously, a 14 percent reduction in the CDS spread from its current level to that of 2014 results in a 1.57 percent ZAR appreciation.

Source: IMF staff.

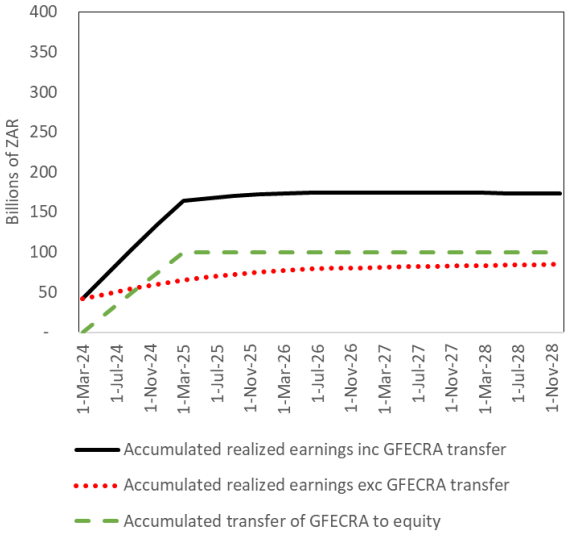
F. Sensitivity Analyses

43. **Two scenarios are examined to see how the policy solvency is impacted by policy decisions:** (i) the impact of constant required reserves; and (ii) a 20 percent increase in the gross foreign reserves. The GFECRA buffer is assumed at 1 percentile and two years. Using sensitivity analyses, we examine how changes in the institutional setting could impact the equity path.

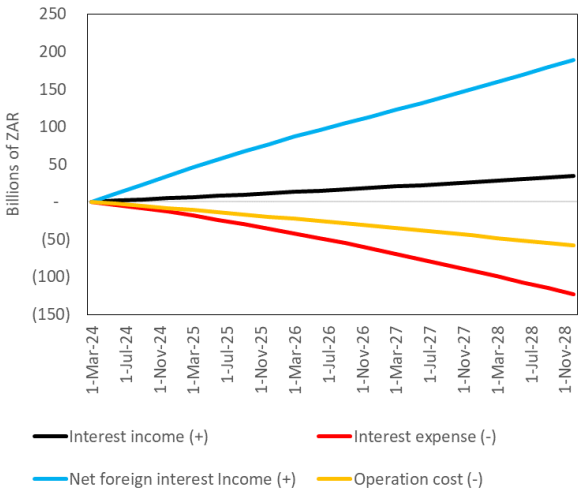
44. **Equity increases under constant required reserves but will be lower than under the base scenario (Figure 12).** Constant required reserves increase the interest expenses on the excess reserves. Recall that the excess reserves decrease by the increase in the required reserves in the core equation. Hence, with no increase in the required reserves, the excess reserves are larger, and the interest expenses are higher. The equity will reach 1.8 percent in 2028Q4 which is 0.1 percent lower than under the base scenario.

Figure 12. Constant Required Reserves

a. Equity



b. Income and Expenses

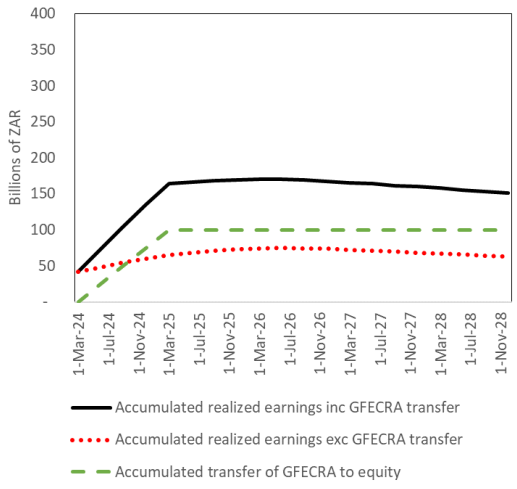


Source: IMF staff.

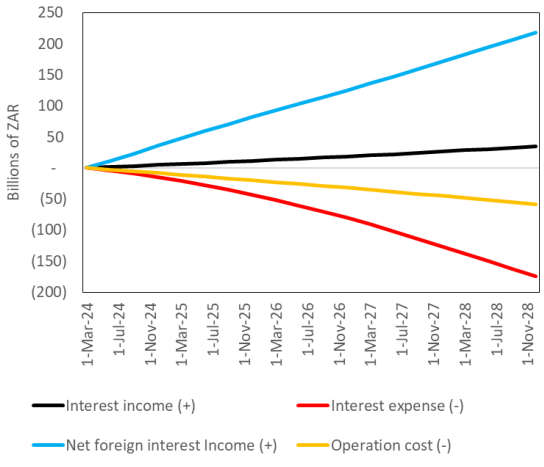
45. **Equity will start to decline in 2026Q1 under the foreign reserve accumulation scenario and will be lower than under the base scenario (Figure 13).** In this scenario, the SARB increases the foreign reserves by 20 percent in USD terms. The accumulation of foreign reserves increases the interest expenses on excess reserves because the excess reserves increase as a result of financing the net USD purchase, as shown in the core equation. Note that foreign interest income also increases, reflecting larger foreign reserves. Yet, the increase in the interest expenses exceeds the increase in the foreign interest income. The equity will reach 1.6 percent in 2028Q4 which is 0.3 percent lower than under the base scenario.

Figure 13. Accumulating Foreign Reserves

a. Equity



b. Income and Expenses



Source: IMF staff.

Annex I. At-Risk Methodology

1. **The mission uses "at-risk" models to forecast conditional distributions for real GDP growth, inflation, and the FX rate ZAR/USD.** Four steps are required to obtain a full distribution of the variable of interest: (i) dimensionality reduction on explanatory variables to avoid overfitting and multicollinearity problems; (ii) conditional average prediction using the Theil-Sen model to accommodate small and "noisy" samples; (iii) skewness estimation of the distribution by the Firth model; and (iv) along with an assumption of Theil-Sen variance, the parameterization of an asymmetrical Gaussian distribution. This method was developed for an IMF Financial Sector Assessment Program mission to the West African Economic and Monetary Union.[1]

2. **The synthetic variables used in the at-risk models are obtained through data reduction based on a set of variables.** The common trend of several variables with the same "theme" (see the complete lists in Tables A1, A2, and A3) are extracted through partial least squares regression.[2] The partial least squares estimator models the covariance between two datasets, named Y and X, based on the latent structure of the underlying data. The latent structure is obtained by projecting both the Y and X matrices on a vectorial lower-dimension subspace, such that the covariance between the projections of Y and X in this new subspace is maximized. The partial least squares method is useful for analyzing data with numerous multicollinear variables that are potentially noisy and may even have incomplete observations. Data reduction through partial least squares is particularly appropriate for aggregating numerous collinear data X, with an objective to maximize the correlation with a supervisor variable Y.

Table A1. Synthetic Regressors and Underlying Variables for Growth At-Risk Model

Domestic Macro	External Demand	Domestic Financial Conditions
Real GDP growth lag	Gold Index (percentage change)	Inflation (first difference)
Private consumption (percentage change)	Crude Oil Index (percentage change)	Policy rate (first difference)
Central government deficit/surplus as percentage of GDP (first difference)	USA real GDP growth	Predominant overdraft rate (first difference)
Gross fixed capital formation (percentage change)	China real GDP growth	Interbank rate (first difference)
Industrial production	REER (percentage change)	3-month CD rate (first difference)
	Import (percentage change)	FTSE/JSE all-share price index (percentage change)
	Export (percentage change)	

Sources: SARB, Bloomberg, Haver, FSI, WEO, and IMF staff calculations.
Note: First difference and percentage change are on a year-on-year basis unless otherwise indicated.

[1] "Financial Sector Assessment Program for West African Economic and Monetary Union: Technical Note on Stress Tests, Credit, Concentration, and Interest Rate Risks." (IMF Country Report No. 22/279).

[2] Wold, S., M. Sjostrom, and L. Eriksson, 2001, "PLS-Regression: A Basic Tool of Chemometrics," *Chemometrics and Intelligent Systems*, 58(2), pp. 109-30.

Table A2. Synthetic Regressors and Underlying Variables for Inflation-at-Risk Model

Domestic Macro	External Demand	Monetary Factor
Real GDP Growth	USD/ZAR (first diff.)	Policy rate (first diff.)
Import (percentage change)	USD/EUR (first diff.)	Predominant overdraft rate (first diff.)
Export (percentage change)	USD/SDR (first diff.)	Interbank rate (first diff.)
FTSE/JSE all-share price index (percentage change)	REER (percentage change)	3-month CD rate (first diff.)

Sources: SARB, Bloomberg, Haver, FSI, WEO, and IMF staff calculations.
Note: First difference and percentage change are on a year-on-year basis unless otherwise indicated.

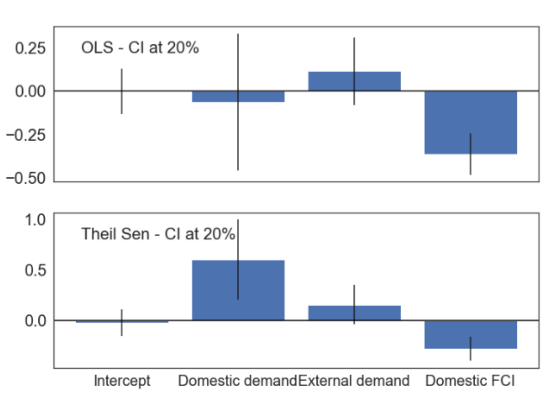
Table A3. Synthetic Regressors and Underlying Variables for Exchange Rate-at-Risk Model

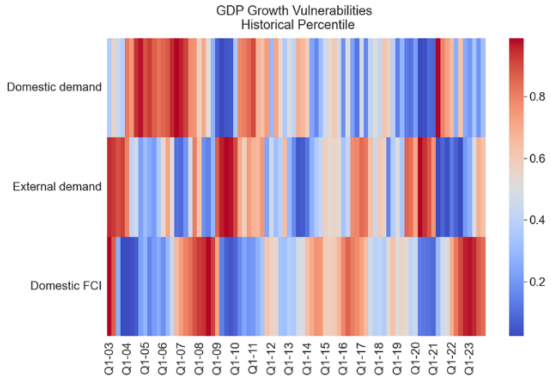
Domestic Macro	Inflation and Interest Rate	Balance of Payment
Real GDP growth	Inflation	Current account balance to GDP (first diff.)
FTSE/JSE all-share price index (percentage change)	Policy rate (first diff.)	Reserves and related items (first diff.)
	Predominant overdraft rate (first diff.)	Financial account balance (first diff.)
	Interbank rate (first diff.)	
	3-month CD rate (first diff.)	

Sources: SARB, Bloomberg, FSI, Haver, WEO, and IMF staff calculations.
Note: First difference and percentage change are on a year-on-year basis unless otherwise indicated.

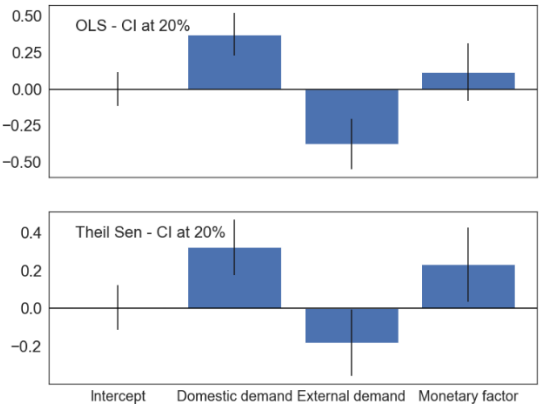
Figure A1. At-Risk Scenarios

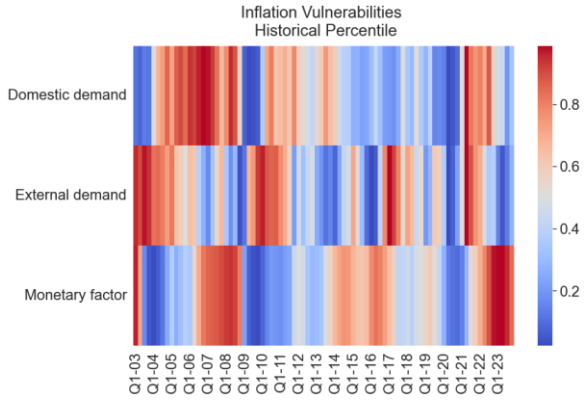
a. Growth-at-Risk



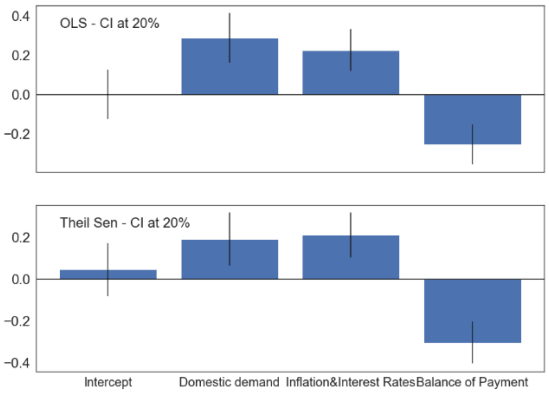


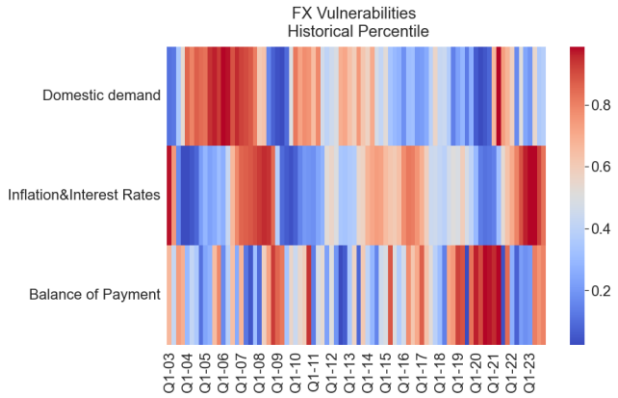
b. Inflation-at-Risk





c. Exchange Rate-at-Risk





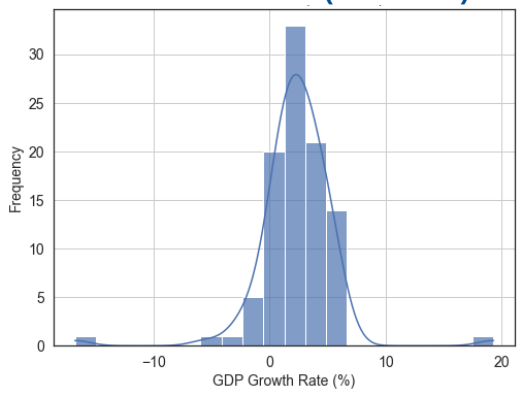
Source: IMF staff calculations.

3. **Growth, inflation, and FX-at-risk scenarios use the 95th percentile (5th percentile for real GDP growth, 5th percentile for FX appreciation) as the worst outcome in forecasting distributions, falling within the historical range.** Figure A2 shows the historical distributions of

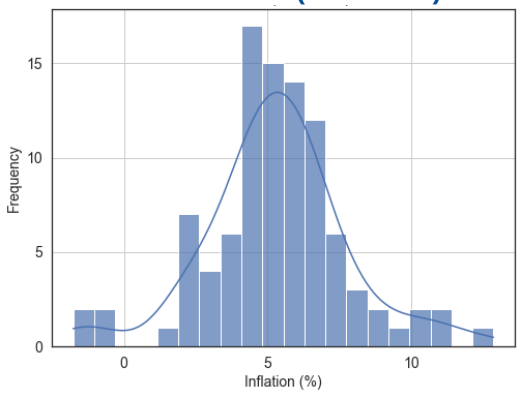
GDP and inflation, indicating that the highest inflation was 13 percent, while the lowest real GDP growth was –17 percent during the period from 2000 to 2023 quarterly. The highest appreciation of ZAR was 31 percent.

Figure A2. Historical Distributions of GDP, Inflation and Exchange Rate

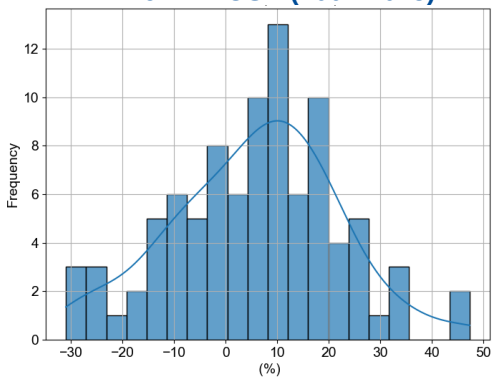
a. GDP Growth (2000-2023)



b. Inflation (2000-2023)



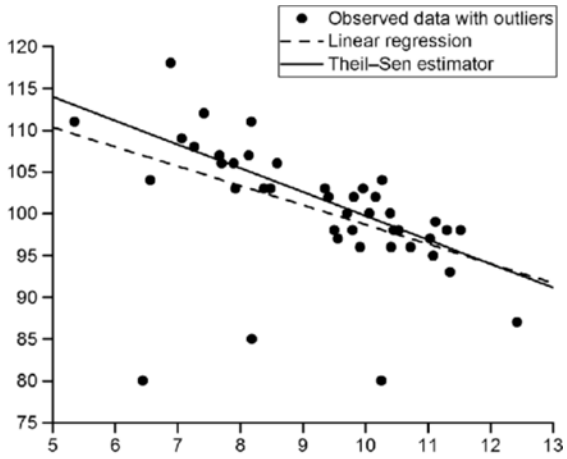
c. ZARUSD (2001-2023)



Source: IMF staff calculation.

4. **The conditional mean is estimated using a Theil-Sen model (Theil 1950; Sen 1968), which is a regression model improving the ordinary least squares estimator to make it more accurate for estimation on small samples and, particularly, robust to outliers.**

Figure A3. The Theil-Sen Estimator and Linear Regression



Source: Scikit-Learn documentation.

A "jackknife" Theil-Sen estimator—one that systemically removes one observation at a time from the initial sample—is constructed. For example, if the sample contains 20 observations, it creates 20 subsamples of 19 observations, with a different observation removed from the original sample at each iteration. It then estimates a classic ordinary least squares regression on each subsample, thus obtaining 20 values for each coefficient. The Theil-Sen estimator is the average of these 20 values, i.e., the average of the ordinary least squares coefficients estimated on each of the subsamples. It is thus highly robust to outliers, to the extent that the impact of such observations is diluted in the estimators for each subsample. Taking the median makes it possible to reduce the impact of coefficients that are too extreme. The specification of the Theil-Sen model is like the one of an ordinary least squares:

$$y_{t+h} = \alpha + \beta^{TS} X_t + \epsilon_t^{TS}$$

Where $y_{t+h}$is the real GDP growth in t+h, X_t is a vector of conditional variables, α is the intercept, and ϵ_t^{TS} is the residuals of the Theil-Sen regression.

5. **The Firth model, a logistic regression model with penalized likelihood, is employed to estimate the asymmetry around the average projection of growth, inflation, or the exchange rate.** Like the classical logistic model, the Firth model estimates the binary probability of an event. This probability may be coded as a binary 0/1 indicator, taking 1 if the event occurs and 0 if it does not. The event is coded as being dependent variable y_t, higher than a given value $\bar{\bar{y}}$. Thus, the specification of the Firth model is written as a classic logistic model.

$$\mathbb{P}[y_{t+h} > \bar{\bar{y}} | X_t] = \alpha + \beta^{LR} X_t + \epsilon_t^{LR}$$

Where $y_{t+h}$is the real GDP growth in t+h, $\bar{\bar{y}}$ is a given growth threshold, X_t is a vector of conditional variables, α is the intercept and ϵ_t^{LR} is the residuals of the logistic regression.

Firth's innovation relies on the estimation method. For small and noisy samples, or samples with a weak degree of separation (a lot of 1 and little 0, for example), the classic logistic estimator is biased. Firth shows that by modifying the likelihood function (the logistic models are estimated

based on maximum likelihood) and by introducing a penalizing term, it is possible to eliminate the estimation bias.[3]

6. **The Theil-Sen/Firth dual model thus estimates two moments in the conditional distribution of the variable of interest, and the third moment is obtained based on a parametric assumption.** The first statistic is the conditional expectation estimated by the Theil-Sen model $\mathbb{E}[\boldsymbol{y_{t0+h}}|X_{t0}] = \hat{\alpha} + \hat{\beta}^{TS}X_{t0}$, while the second is the asymmetry of the distribution, obtained as the cumulative density estimated at the conditional mean[4] $\mathrm{F}(y_{t0}|X_t) = \hat{\alpha} + \hat{\beta}^{LR}X_{t0}$. These two statistics are not sufficient to parameterize a distribution, as the second order moment is missing, i.e., the variance. Estimating the conditional variance on a limited sample is discouraged, as the estimators of conditional variance need a lot of information to estimate heteroskedasticity (as in the case of an ARCH/GARCH model, for example). Thus, the at-risk model makes the simplifying but realistic assumption that the variance is unconditional and equal to the residual variance of the Theil-Sen estimation (i.e., heteroskedasticity is assumed to be constant over the course of time). This approach also addresses a recurring problem of projection models, i.e., that the variance of the projection tends to increase with the projection's horizon. With constant heteroskedasticity, there is no inflation in the variance. Thus, under this assumption, the at-risk model obtains three conditional moments: the expectation (Theil-Sen projection), the variance (constant heteroskedasticity, derived from Theil-Sen), and the skewness (obtained from the Firth logistic model).

7. **The team parameterizes an asymmetrical Gaussian distribution from the three estimated moments.** The at-risk model further stabilizes the projection by using an overparameterized fit, where the distribution is assumed to follow an asymmetric Gaussian process. This assumption is realistic, insofar as an asymmetric Gaussian distribution naturally encompasses both the standard normal distributions and the asymmetric ones. This approach retains a high degree of generality while conserving simplicity. It presents the most interesting metrics for economists (central tendency, interquartile range, and balance of risks). The choice of an asymmetrical Gaussian, rather than another asymmetrical distribution, is constrained by the number of moments. To estimate an asymmetric Student distribution, four moments are needed (including the kurtosis), which, due to the limited size of the sample, is unfeasible. Another approach consists of using nonparametric distributions, like kernels, but again, the limited size of the samples makes this approach unsuitable. Finally, a major advantage of the asymmetric Gaussian distribution is that it provides simple analytical relationships between moments, cumulative density, and parameters. This property greatly simplifies the distribution fit on conditional moments, as the parameters are derived manually in closed algebraic form and not through optimized approximation.

[3] Firth, D., 1993, "Bias Reduction of Maximum Likelihood Estimates," *Biometrika*, 80(1), pp. 27–38.

[4] This quantity is not directly a measure of asymmetry. However, in the case of an asymmetrical Gaussian distribution, it is possible to infer the asymmetry coefficient from $\mathrm{F}(y_{t0}|X_t)$ via a simple bijective transformation.