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No-Arbitrage One-Factor Models of the South African Term-Structure of Interest Rates

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Abstract

Short-term interest rate processes determine the term-structure of interest rates in an arbitrage-free market, and are central to the valuation of interest-rate derivatives. We obtain parameter estimates and compare the empirical fit of alternative one-factor continuous-time processes for the South African short-term interest rate (and hence of arbitrage-free term-structure models), using Gaussian estimation methods. We find support only for diffusions where the interest rate volatility is moderately sensitive to the level of the interest rate – with particularly clear results after the adoption of inflation targeting. Other common models with restrictions that either preclude this effect, or restrict it to be too high, do not fit the data. Differences in the specification of the drift function have no evident effect on model performance.

JEL classification: G12, C13, E43.

Keywords: continuous-time short-rate models; bonds and interest rate derivatives; arbitrage theory; maximum likelihood; term-structure of interest rates.

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1 Introduction

The short-term interest rate, and its behavior over time, is essential for the relative pricing of bonds and other interest-rate derivatives; for interest-rate risk management; and as a benchmark for measuring the cost of capital in the economy. The relationship between short-term rates (or its dynamics) and long-term rates is also an important aspect of the monetary transmission mechanism. The classic expectations hypothesis about the term-structure of interest rates remains an important and commonly used result in macro-economics and monetary policy. (Guidolin and Thornton (2008), Fedderke and Pillay (2010).) Generally put, it states that long-term rates are a weighted average of expected (or forecast of) short-term rates, under a zero or constant risk premium.

The expectations hypothesis is intuitively appealing, and easy to implement. However, as originally shown by Cox, Ingersoll, and Ross (1981) three decades ago, there are in fact at least three different formulations for the expectations hypothesis; these are mutually contradictory, except under highly restrictive conditions; and only one version, applicable only for one holding period (the "local expectations hypothesis"), is consistent with rational expectations equilibrium in continuous time. Traditional forms of the expectations hypothesis generate term structures of interest rates that may not rule out arbitrage opportunities in the bond market − a highly undesirable feature given the activities of investment banks and hedge funds in international financial markets. The voluminous empirical evidence is also largely inconsistent with the traditional expectations hypotheses. (See for example Ang, Dong, and Piazzesi (2005), and Guidolin and Thornton (2008) for a recent review.)

The more recent approach to modelling the term-structure of interest rates focuses on the arbitrage-free pricing of fixed-income securities.[1] Most

[1] The seminal contributions include Cox (1975), Vasicek (1977), Dothan (1978), Bren-

models in the arbitrage approach were developed in a continuous-time setting, where a diffusion is specified for the evolution of the short-term interest rate, which in turn is intimately connected to the term-structure through an arbitrage argument, as further explained below. The stochastic behavior of short-term rates in these models is consistent with evidence (in for example Guidolin and Thornton (2008)) that short rates are largely unpredictable at relatively high frequencies. The continuous-time setting, in addition to immense analytic convenience, is consistent with the high frequency of changes in market rates.

This paper provides an econometric examination of commonly used stochastic models of the short-term interest rate process. Specifically, we provide an empirical comparison of alternative non-linear single-factor continuous-time models for South African short-term interest-rate dynamics using Gaussian (maximum likelihood) estimation methods. We employ a discretization scheme due to Bergstrom (1983, 1984, 1985, 1986, 1990) and introduced to the interest rate modelling literature by Nowman (1997).

South Africa's fixed-income market is one of the largest among emerging markets, and its government bond market is the world's sixth most liquid by turnover (Bank for International Settlements (2007), p. 45).[2] Despite the current size and liquidity of the South African fixed income market, an econometric examination of continuous time diffusions for the South African short-rate, or any other aspect of the arbitrage approach applied to the South African fixed income market, were hitherto inexistent, to the authors' knowledge.

Our results are consistent with Chan, Karolyi, Longstaff and Sanders (1992) and the subsequent literature (see for example Tse (1995) and Nowman (1997, 1998)), in finding that the sensitivity of interest rate volatility to the level of the interest rate is the central feature in differentiating continuous-time interest rate models. Among standard models for short-term interest rate dynamics, diffusion models which allow the volatility of interest rates to be a function of the level of the interest rate (a "level effect"), and restrict this sensitivity to one, provide the best empirical fit for South African data, in the period following the introduction of inflation

nan and Schwartz (1980), Cox, Ingersoll, and Ross (1980, 1985), and Heath, Jarrow, and Morton (1992). This list is far from exhaustive however, and the literature continues to expand, through contributions from researchers in academia as well as the financial sector - see for example Jamshidian (1997), Brace, Gatarek and Musiela (1997), James and Webber (2000) .

[2]See also the press article by Bonorchis, "Global Bank Praises SA Bond Market" in the *Business Day*, 10 September, 2007.

targeting.[3] For the prior period, there is only support for the Constant Elasticity of Variance diffusion (Cox (1975)), which does not restrict the magnitude of the level effect. Other standard specifications provide a very poor fit with the data.

The remainder of the paper proceeds as follows. Section 2 presents a compact treatment of the arbitrage-free term-structure equation, and explains the relationship between short-rate models and the term-structure of interest rates. Section 3 discusses related literature. Section 4 describes the system of single-factor continuous-time models estimated, and explains the econometric method. Section 5 describes the data, provides a brief overview of policy changes relevant to the South African fixed-income market, and presents summary statistics. Section 6 presents the empirical results for the full sample period, and two sub-samples, separated by the first target year after the adoption of inflation targeting in South Africa. Section 7 concludes.

2 Theoretic background: the no-arbitrage term-structure equation

The central aim of the arbitrage approach to interest rate theory (simply interest rate theory, henceforth), is to explore the relationship between fixed-income securities prices in an arbitrage-free world. Arguably the most fundamental object in this framework, is the no-arbitrage term-structure of interest rates equation. Its role in interest rate theory is equivalent to that of the Black-Scholes equation (Black and Scholes (1973)) in general arbitrage theory; and its derivation is a variation on the now standard Black-Merton-Scholes arbitrage argument. (Black and Scholes (1973), Merton (1973).) The term-structure equation summarizes the relationship that must hold between prices of bonds of different maturities in the absence of arbitrage opportunities in the fixed income market, and, by extension, arbitrage-free prices of any interest rate derivative - relative to a benchmark bond. What follows is a compact treatment of the term-structure equation and its solution. The aim of this section is to clarify how each of the continuous-time models for the short-term interest rate is intimately connected to a specific solution to the term-structure equation; and hence, modelling the short-rate dynamics is equivalent to modelling the term-structure of interest rates. For more complete treatments see for example the excellent expositions in Björk

[3] The standard models examined are Merton (1973), Vasicek (1977), Dothan (1978), Brennan and Schwartz (1980), Cox, Ingersoll and Ross (1980, 1985), as well as standard geometric Brownian motion and the Cox (1975) constant elasticity of variance diffusion.

(1997, 2004) and Duffie (2001), on which this summary is based.

Let $P(t, T)$ denote the price at time t of a zero-coupon bond with maturity at time T, and terminal payoff normalized to 1.[4] There is a (locally) risk-free asset, with price B and dynamics

$$dB_t = r_t B_t dt, \tag{1}$$

with dynamics of r (the short-rate) given by the stochastic differential equation:

$$dr_t = \mu_t dt + \sigma_t dW_t, \tag{2}$$

where W is a Brownian motion under probability measure P, and μ_t and σ_t are, respectively, the drift and diffusion functions. We can regard $P(t, T)$ as a stochastic object with two variables, t and T. If we fix t, then $P(T; t)$ is the term-structure at t; if we fix T, then $P(t; T)$ is a scalar process, giving prices, at different points in time, of a bond with fixed maturity T. The rate r is the continually compounding rate of interest on a risk-free (i.e. deterministic dynamics) security, or bank account process.

To apply arbitrage pricing, we take the price of one particular benchmark bond as given, and determine the arbitrage-free prices of all other bonds (or interest rate derivatives) in terms of the price of the benchmark, and the assumed dynamics for r. Specifically, let $P(t, T) = F^T(t, r)$, and $P(t, S) = F^S(t, r)$ where $F^T, F^S \in C^{n,m}$. The rest of the analysis leading to the term-structure equation is standard (see the Appendix): apply Itô's formula to obtain the dynamics of $P(t, T)$ and $P(t, S)$; form a portfolio of the T- and S-bonds; and choose the portfolio weights so that the portfolio has deterministic dynamics (i.e. it is locally risk-free). In the absence of arbitrage opportunities, such a portfolio cannot earn more (nor less) than the risk-free asset. Equating the portfolio returns to the risk-free return gives the term-structure equation:

$$\frac{\partial F^T}{\partial t} + \frac{\partial F^T}{\partial r}(\mu_t - \lambda_t \sigma_t) + \frac{1}{2}\frac{\partial^2 F^T}{\partial r^2}\sigma_t^2 - rF^T = 0, \tag{3}$$

with terminal condition $F^T(T, r) = 1$ (for zero-coupon bond pricing and the term-structure of interest rates), and where λ reflects the risk premium in the bond market.

We now wish to emphasize the connection between the short-rate dynamics and the solution to the term-structure equation − and, by extension, the pricing of bonds and interest-rate derivatives. Girsanov's theorem ensures

[4] Coupon-paying bonds are simple portfolios of zero-coupon bonds.

the existence of a probability measure Q (commonly known as a martingale measure), equivalent to P, such that for any process θ, $\widetilde{W_t}$ defined by

$$\widetilde{W_t} = W_t + \int_0^t \theta_s ds, \tag{4}$$

is a Brownian motion under Q.[5] Technical conditions aside, from an economic viewpoint the existence of a martingale measure is equivalent to the absence of arbitrage opportunities. (Harrison and Kreps (1979).) Let $\theta_t = \frac{\mu_t - (\mu_t - \lambda_t \sigma_t)}{\sigma_t}$, and substitute 4 (in differential form) into equation 2. This gives us the r-dynamics under Q:

$$dr_t = (\mu_t - \lambda_t \sigma_t)\, dt + \sigma_t d\widetilde{W_t}, \tag{5}$$

where $\widetilde{W}$ is a Brownian motion under Q. The Feynman-Kač stochastic representation theorem (Duffie (2001), p.93-94,139, 342-343; Björk (2004), p. 68-72) gives the bond price solution to the term-structure equation in probabilistic form as

$$F^T(t, r) = E^{\mathbf{Q}}_{t,r}\left[\exp\left(-\int_t^T r_s ds\right) \times 1\right], \tag{6}$$

where r satisfies equation 5.

For more general interest rate claims, simply change the terminal condition to $F^T(T, r) = \phi(r)$, where $\phi(r)$ is the contract function specifying the derivative's payoff at maturity. The notation used for the expectation operator in equation 6, emphasizes the dependence of the solution to the term-structure equation, obtained as a discounted expectation under a martingale measure Q, on the probability law implied by the r-dynamics. In other words, different specifications of the short-term interest rate dynamics will result in different solutions to the term-structure equation. This point is made explicit by equation 6.

To summarize: in interest rate theory, the price of a zero-coupon bond with normalized payoff of given maturity is equal to its discounted expectation (at the risk free rate) under a martingale measure, the existence of which is guaranteed by the absence of arbitrage, and which is associated with the risk premium in the bond market. (See the appendix.) This expectation solves the term-structure equation that must be satisfied by the

[5] The result is subject to a technical condition on the Girsanov kernel θ (the Novikov condition), easily satisfied in finance applications. See Duffie (2001), p.111, 337-338, Björk (2004), p.160-162, 323-324.

price of any interest rate claim in the absence of arbitrage opportunities in the fixed income market. The probability law we use to solve this expectation, if an analytic solution exists, depends on the specific stochastic differential equation used to describe the evolution of the short-rate (a default risk-free instantaneous yield). Thus, subject to a suitable change of probability measure, associated with a given description of the short-rate dynamics is a specific arbitrage-free term-structure of interest rates. This relationship makes the choice of model for short-rate dynamics central for the arbitrage-free modelling of the term-structure, pricing of fixed income securities (interest rate derivatives in particular), and management of interest rate risk.

Which short-rate model should be used in a given application is a practical question. Empirical analysis can help us restrict the set of suitable models for a given market − especially regarding the treatment of the diffusion (volatility) term.

3 Related literature

Chan, Karolyi, Longstaff and Sanders (1992) (henceforth CKLS) is a contribution of reference to the econometric estimation of single-factor continuous-time short-rate models. They propose a general representation of continuous-time interest rate dynamics, which nests a range of standard models as special cases, and use the Euler method to obtain a discrete-time approximation of the continuous-time system. The generalized method of moments (GMM) technique (Hansen (1982)) is then applied to obtain parameter estimates and compare the empirical fit of competing specifications for the United States short-term rate. Tse (1995) applies the same method to examine the short-rate processes for a group of advanced economies; Brailsford and Maheswaran (1998) apply it to Australian rates, and McManus and Watt (1999) to the Canadian term-structure.

It is now known however that, except for exceptionally large samples (in the region of over one and a half thousand observations), the linear approximation proposed in CKLS introduces a discretization bias, due to temporal aggregation, resulting in inconsistent estimators. (Melino (1994), Baadsgaard, Nielsen, Madsen and Preisel (1996), Yu and Phillips (2001).)

To correct for the bias in the CKLS discretization, Nowman (1997) proposed an application of the method developed by Bergstrom (1983, 1984, 1985, 1986, 1990), whereby an exact discrete-time model is used as the basis for Gaussian estimation of the parameters of a continuous-time model –

taking into account the restrictions on the distribution of the discrete-time data implied by the continuous-time model. The general specification in CKLS allows for non-constant volatility, which impedes direct application of Bergstrom's method. As a solution, Nowman (1997) assumed that the volatility of the interest rate process changes at the beginning of each unit observation period, but stays constant over the unit interval. The associated approximation produces a conditional Gaussian distribution for the short-rate, from a non-Gaussian process. The parameters can be estimated by maximum likelihood.

Yu and Phillips (2001) develop an alternative approach to forming a discrete-time model of continuous-time short-rate models, with Gaussian errors. Their contribution stems largely from the alternative estimation of the drift parameters, since the method relies on the Nowman (1997) method to estimate the diffusion terms. From an applications viewpoint however, the main benefit of econometric estimation of continuous-time processes is the estimation of the diffusion parameters. Estimates of drift parameters, obtained through application of statistical methods to real-world data, can only be used for bond and derivative pricing if the bond market risk premium is known, since we are characterizing the short-rate dynamics under the data-generating measure; but solve the arbitrage-free term-structure equation using the short-rate probability law under a risk-neutral measure.[6] Lastly, the method, which involves a time-transformation requiring unequally spaced observations, would run into implementation difficulties in South Africa's relatively high interest rate environment, due to the need to sample the process more frequently when interest rates are high.

4 Short-rate models and econometric method

4.1 Continuous-time short-rate models

Consider the following stochastic differential equation for the dynamics of the short-rate,

$$dr_t = (\alpha + \beta r_t)\,dt + \sigma r_t^{\gamma} dW_t, \tag{7}$$

[6] Girsanov's theorem connects the short-rate dynamics under the data-generating probability measure, and the dynamics under a martingale measure, through the market price of risk, which pins down the Girsanov kernel. See Duffie, p111 and 337-338, or Björk (2004), Proposition 21.4, p.323-324, Theorem 11.3, p. 160-161, and Remark 11.3.2, p. 162.

where r_t is a stochastic interest rate process, W_t is a standard Brownian, and α, β, γ and σ are unknown structural parameters.[7] In this specification the interest rate reverts to its unconditional mean -$\frac{\alpha}{\beta}$, and β is the speed of reversion to the mean. Notice that both the drift and the conditional variance of the process are functions of the level of the interest rate. The parameter γ measures the sensitivity of the variance to the level of the interest rate.

Table 1 presents a list of standard continuous-time interest rate models, in the order used in CKLS (with an increasing level effect). Each of the diffusion models in the second column can be obtained from equation 7 by imposing appropriate restrictions on parameters α, β and γ (none of the models impose restrictions on σ). The associated restrictions are shown in the last three columns of Table 1. The first model is a simple "arithmetic" Brownian motion with a constant drift parameter α, used by Merton (1973) to obtain no-arbitrage prices of zero-coupon bonds. It is analytically a very simple model, but the interest rate can be negative, and both the drift and volatility of interest rates are constant. The Vasicek (1977), CIR (1985) and Brennan and Schwartz (1980) models permit mean-reversion in the interest rate process - i.e. higher (resp., lower) short-rates leading to lower (higher) drift. Vasicek (1977) is an Ornstein-Uhlenbeck process, with the short-rate as an auto-regressive process of order 1; CIR (1985) implies a non-central χ^2 distribution for interest rate changes. The Dothan (1978) and CIR (1980) specifications imply no drift in the interest rate process. GBM is the process used in Black and Scholes (1973), and widely applied to price simple stock options. It implies a log-normal distribution for interest rates.

[7] With reference to equation 2, here $\mu_t = (\alpha + \beta r_t)$, and $\sigma_t = \sigma r_t^{\gamma}$.

Table 1

Parameter Restrictions Imposed by Alternative Short-Rate Models

	Model	α	β	γ
Merton (1973)	$dr_t = \alpha dt + \sigma dW_t$		0	0
Vasicek (1977)	$dr_t = (\alpha + \beta r_t)\, dt + \sigma dW_t$			0
CIR$^\sharp$ (1985)	$dr_t = (\alpha + \beta r_t)\, dt + \sigma r_t^{1/2} dW_t$			1/2
Dothan (1978)	$dr_t = \sigma r_t dW_t$	0	0	1
GBM†	$dr_t = \beta r_t dt + \sigma r_t dW_t$	0		1
Brennan-Schwartz (1980)	$dr_t = (\alpha + \beta r_t)\, dt + \sigma r_t dW_t$			1
CIR$^\sharp$ (1980)	$dr_t = \sigma r_t^{3/2} dW_t$	0	0	3/2
CEV‡	$dr_t = \beta r_t dt + \sigma r_t^\gamma dW_t$	0		

($\sharp$): Cox, Ingersoll and Ross
($\dagger$): Geometric Brownian Motion
($\ddagger$): Constant Elasticity of Variance

These are commonly used single factor models for continuous-time short-rate dynamics. Perhaps the most noteworthy difference between the alternative specifications concerns the modelling of volatility. This is of practical importance because short-rate volatility is a crucial input for the management of interest rate risk. The first two models, Merton (1973) and Vasicek (1977), treat volatility as constant. As we move down the list, short-rate volatility becomes increasingly sensitive to the level of the interest rate. The constant elasticity of variance model (Cox (1975)), henceforth CEV, does not restrict the magnitude of this sensitivity.

4.2 The Bergstrom-Nowman Gaussian method

The stochastic integral for equation 7 is

$$r_T = r_t + \int_t^T (\alpha + \beta r_s)\, ds + \sigma \int_t^T r_s^\gamma dW_s. \tag{8}$$

Suppose we fix the volatility of the short-rate at the beginning of the unit observation period, so that, over $[\tau, \tau + 1)$, r_t has dynamics

$$dr_t = (\alpha + \beta r_t)\, dt + \sigma r_\tau^\gamma dW_t, \tag{9}$$

where $\tau \leq t < \tau + 1$. The stochastic integral is given by

$$r_t = r_\tau + \int_\tau^t (\alpha + \beta r_s)\, ds + \sigma r_\tau^\gamma \int_\tau^t dW_s. \tag{10}$$

Bergstrom (1984) gives the corresponding exact discrete model as

$$r_t = e^{\beta} r_{t-1} + \frac{\alpha}{\beta}\left(e^{\beta} - 1\right) + \eta_t, \qquad (t = 1, 2, ..., T) \tag{11}$$

with conditional distribution $\eta_t \mid \mathcal{F}_{t-1} \sim N\left(0, m_t^2\right)$, where

$$m_t^2 = \frac{\sigma^2}{2\beta}\left(e^{2\beta} - 1\right) r_{t-1}^{2\gamma}. \tag{12}$$

The Bergstrom-Nowman method is applied to equation 11. Let θ denote the parameter vector $\theta = \left(\alpha, \beta, \gamma, \sigma^2\right)$. Given the distribution of η_t, the log-likelihood function for 11 is given by

$$L(\theta) = -\frac{1}{2}\sum_{t=1}^{T}\left[\log\left(2\pi m_t^2\right) + \frac{\eta_t^2}{m_t^2}\right], \tag{13}$$

where T is the total number of observations. The estimates of the model parameters are found as $\widehat{\theta} = \arg\max_{\theta}\{L(\theta)\}$.

For comparison, we also obtain Gaussian estimates using the CKLS discrete approximation of equation 7, given by:

$$r_{t+1} - r_t = \alpha + \beta r_t + \eta_{t+1}, \tag{14}$$

where

$$E_t\left(\eta_{t+1}\right) = 0 \text{ and } E_t\left(\eta_{t+1}^2\right) = \sigma^2 r_t^{2\gamma}. \tag{15}$$

5 Data, policy environment and descriptive statistics

5.1 Data and policy environment

The short-rate process used in no-arbitrage interest rate theory is an abstract object, denoting an instantaneous rate that has no direct empirical equivalent. The natural approach is to use default-risk free fixed income contracts with the lowest maturity available. This is normally the overnight rate. However, the determinants of the overnight rate can differ from the forces that drive longer rates. Hence, the overnight rate can be insufficiently closely correlated with other fixed-income market rates. In single-factor models, in contrast, yields for bonds of different maturities are perfectly correlated. This is a common assumption, though not strictly accurate, in bond risk management; and a valid assumption when pricing derivatives subject to one source of risk. (See, for example, James and Webber (2000).) The widely used alternatives in empirical research are one- and three-month Treasury bill rates. For example, CKLS use one-month bills; Tse (1995), Brenner, Harjes and Kroner (1996), and McManus and Watt (1999), use three-month bills.

We use the rate on the South African three-month (91 days) Treasury bill, which is commonly used as a market-determined proxy for the domestic short-term risk-free rate (e.g., Fedderke and Pillay (2010), Hassan and Van Biljon (2010)), obtained from the South African Reserve Bank. The data are weekly, taken on the Monday of each week, covering the period from the 18th of June 1984, to the 18th of July 2011, giving a total of 1322 observations. Figure 1 shows the evolution of the level of the Treasury bill rate over the 1984-2011 period; Figure 2 shows the evolution of short-term changes in the three-month interest rate over the same period.

Figure 1

South African Three-Month Treasury-Bill Rate: 1984 – 2011

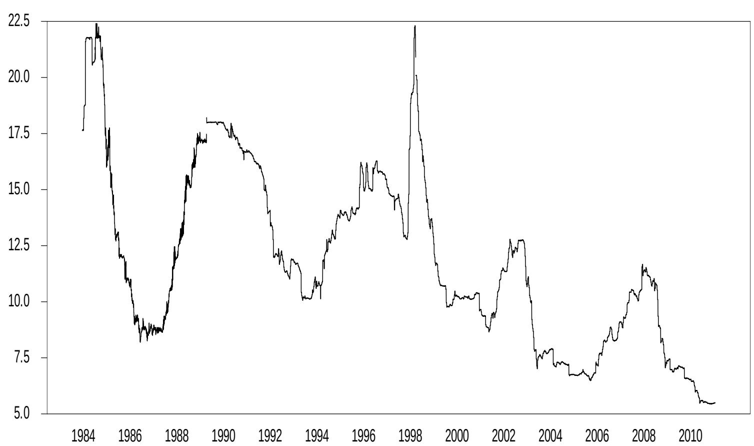
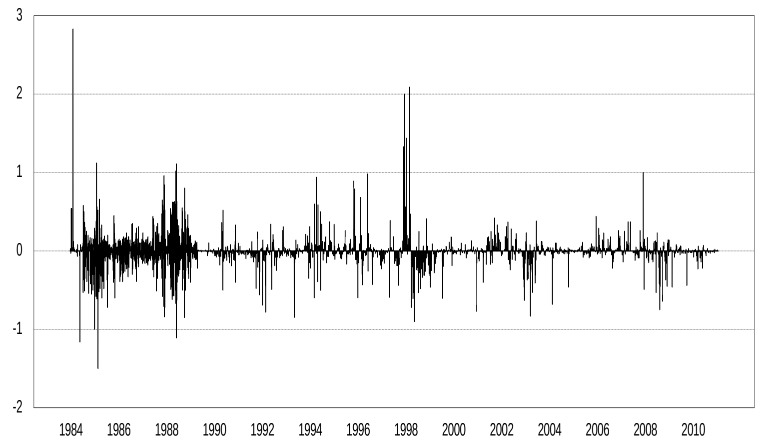


Figure 2

First Differences of Three-Month Treasury-Bill Rate: 1984 – 2011



We start the sample in the mid-1980s due to the number of policy changes and extent of Government intervention in the interest rate market until then. The prevailing regime between 1957 and the early 1980s was a liquid asset ratio-based system with quantitative controls on interest rates and credit. This was gradually reformed toward a cash reserves-based system. Pre-announced, flexible monetary target ranges were used from 1986, with the main policy emphasis on the central bank's discount rate in influencing the cost of overnight collateralized lending and hence market interest rates (Aron and Muellbauer (2001)). In addition, recent research indicates that South Africa's capital controls permitted the South African Reserve Bank to target domestic interest rates through interventions in the foreign exchange market (Schaling (2009)). The monetary authorities only began adopting a more market-oriented policy environment in the late 1970s (Farrell and Todani (2004)). Lastly, and arguably at least partly as a result of previous policy, there was virtually no active secondary market for trading in government securities in South Africa until 1982 (McLeod (1990)).

The exceptionally high interest rate volatility in the mid to late 1980s reflects the country's political instability during the pre-1994 political dispensation. The level and magnitude of up and down movements in the interest rate decrease gradually from the late 1990s. In 2000, the South African monetary authorities adopted inflation targeting as policy, with 2002 as the first target year. This policy intervention may affect short-term interest rate dynamics and indicates a natural sample split. We present results for the full sample period; and separately for the periods before and after inflation targeting.

5.2 Descriptive statistics

Table 2 shows the summary statistics for the full sample period, and the two sub-samples. It shows the means, standard deviations and autocorrelations of the South African three-month Treasury bill rate, and the first differences for the same series. The variable $r(t)$ denotes the level of the interest rate, and $\Delta r(t)$ is the weekly change in r. T represents the number of observations used; SD is the standard deviation; ρ_j denotes the autocorrelation coefficient of order j; ADF denotes the Dickey-Fuller-Saïd (Saïd and Dickey (1984)) unit root statistic, or Augmented Dickey-Fuller, with a five percent critical value of -2.860.

Table 2

Summary Statistics

	T	Mean	SD	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ADF
				A. Full sample period: 1984-2011					
$r(t)$	1322	11.909	3.94	0.94	0.93	0.92	0.92	0.92	-1.12
$\Delta r(t)$	1321	–0.009	0.24	0.22	0.15	0.14	0.05	0.07	-27.17
				B. Period prior to inflation targeting: 1984-2002					
$r(t)$	865	13.733	3.44	0.95	0.94	0.93	0.92	0.92	-1.00
$\Delta r(t)$	864	-0.012	0.27	0.21	0.15	0.15	0.06	0.05	-22.39
				C. Period under inflation targeting: 2002-2011					
$r(t)$	457	8.457	2.05	0.92	0.90	0.89	0.88	0.88	-0.06
$\Delta r(t)$	456	-0.005	0.14	0.30	0.12	0.10	0.06	0.13	-14.23

T is the number of (weekly) observations; SD is the standard deviation.
Means and standard deviations are in percentage terms.

The inflation targeting period is associated with a much lower average level, and lower volatility of the Treasury bill rate. The average rate of interest on three-month bills reduced from 13.7 to 8.4 percent; its standard deviation reduced from 3.4 to 2 percent. The autocorrelations of the level variable fall off slowly, whilst the autocorrelations of the first differences are small and neither systematically positive nor negative. This indicates the presence of a unit root, confirmed by the ADF statistic which fails to reject the null hypothesis of a unit root at the 5 percent level of significance.

6 Results

We present the Gaussian estimation results from the unrestricted model and the eight nested term-structure models obtained after imposing the appropriate restrictions on the general model. We contrast the fit of the different models to the unrestricted model by comparing the maximized Gaussian likelihood function values, and performing likelihood ratio tests. Each table shows the Gaussian coefficient estimates; their standard errors; maximized log likelihoods for the unrestricted and eight nested models; and the likelihood ratio tests comparing the nested models with the unrestricted model.

6.1 Results for the full period

Table 3 reports the results using the entire series. Based on the maximized Gaussian likelihood values, compared with that of the unrestricted model, the CEV model performs best, followed by the CIR (1985), Brennan-Schwartz, GBM and Dothan models. All the best performing models include a γ coefficient greater than zero - indeed, equal to or greater than one, except for the CIR (1985) model. This finding suggests that the conditional volatility is dependent on the level of the interest rate - a "level effect". However, using the χ^2 likelihood ratio test under the null hypothesis that the nested model restrictions are valid, we can reject the Merton, Vasicek, CIR (1985), Dothan, GBM, Brennan-Schwartz and CIR (1980) models. We only fail to reject the CEV model.

Both the unrestricted and CEV models estimate γ at 0.743, and the estimates are statistically significant. There is no clear evidence of a linear trend: in all models, estimates of α are very close to zero, negative for Merton and Brennan-Schwartz, positive for the others. There is only weak evidence of mean-reversion: most models produce negative estimates of β, but these are very close to zero (we expect $\alpha > 0$, $\beta < 0$). The parameters α and β are not statistically significant in the unrestricted model.

Interestingly, observe that the asymptotic bias resulting from the CKLS approximation is very small. The estimates are almost identical under the CKLS and Bergstrom-Nowman approaches.

Table 3

Gaussian Estimates of Continuous-Time Models of the Short-Rate, 1984-2011

Full sample, 06/1984 - 07/2011

Model	α	β	σ^2	γ	Log Likelihood	χ^2 Test	Df
Unrestricted	0.0022499	-0.0009125	0.0052775*	0.743457*	273.57399		
	(0.0139)	(0.0015249)	(0.0008224)	(0.0318389)			
CKLS	0.0022489	-0.0009121	0.0052727*	0.7434571*	273.57399		
	(0.0138921)	(0.0015235)	(0.0008214)	(0.0318389)			
Merton	-0.009808	0.0	0.2406477*	0.0	6.853248	533.44	2
	(0.0068065)		(0.004813)			(< 0.0001)	
CKLS	-0.009808	0.0	0.2406477*	0.0	6.8532478	533.44	2
	(0.0068065)		(0.004813)			(< 0.0001)	
Vasicek	0.0132824	-0.0019315	0.2409914*	0.0	7.4831682	532.18	1
	(0.0216835)	(0.001722)	(0.0048376)			(< 0.0001)	
CKLS	0.0132695	-0.0019296	0.2405265*	0.0	7.4831682	532.18	1
	(0.0216516)	(0.0017187)	(0.0048105)			(< 0.0001)	
CIR SR	0.0052371	-0.0012244	0.0176116*	0.5	244.22069	58.71	1
	(0.0155273)	(0.001536)	(0.0003533)			(< 0.0001)	
CKLS	0.0052339	-0.0012237	0.01759*	0.5	244.22069	58.71	1
	(0.015514)	(0.0015341)	(0.0003518)			(< 0.0001)	
Dothan	0.0	0.0	0.0015592*	1.0	240.55416	66.04	3
			(0.0000312)			(< 0.0001)	
CKLS	0.0	0.0	0.0015592*	1.0	240.55416	66.04	3
			(0.0000312)			(< 0.0001)	
GBM	0.0	0.0006526	0.0015589*	1.0	241.62514	63.90	2
		(0.0004459)	(0.0000312)			(< 0.0001)	
CKLS	0.0	0.0006524	0.0015578*	1.0	241.62514	63.90	2
		(0.0004456)	(0.0000312)			(< 0.0001)	
Brennan-Schwartz	-0.000964	-0.0005411	0.0015587*	1.0	241.62794	63.89	1
	(0.0128688)	(0.0015542)	(0.0000313)			(< 0.0001)	
CKLS	-0.0009638	-0.0005409	0.015578*	1.0	241.62794	63.89	1
	(0.012866)	(0.0015534)	(0.0000312)			(< 0.0001)	
CIR VR	0.0	0.0	0.0001658*	1.5	9.3623157	528.42	3
			(0.00000332)			(< 0.0001)	
CKLS	0.0	0.0	0.0001658*	1.5	9.3623157	528.42	3
			(0.00000332)			(< 0.0001)	
CEV	0.0	-0.0006778	0.0052729*	0.7435914*	273.56088	0.03	1
		(0.0004734)	(0.0008212)	(0.0318305)		(0.8714)	
CKLS	0.0	-0.0006776	0.0052693*	0.7435914*	273.56088	0.03	1
		(0.004731)	(0.0008206)	(0.0318305)		(0.8714)	

6.2 Results for the period prior to inflation targeting

The results are qualitatively very similar for the sample period prior to inflation targeting (865 observations, from 18 June 1984 to 28 January 2002), which represents two-thirds of the full sample. We find the same pattern of results, with a marginally lower level effect ($\gamma = 0.71$ for the unrestricted model), and the CEV model offering the best fit with the data by a wide margin.

Table 4
Gaussian Estimates of Continuous-Time Models of the Short-Rate
Before inflation targeting, 06/1985 - 01/2002

Model	α	β	σ^2	γ	Log Likelihood	χ^2 Test	df
Unrestricted	0.0266469	-0.0028382	0.0058842*	0.7197319*	-7.9214722		
	(0.0322502)	(0.0027097)	(0.0014955)	(0.0488345)			
CKLS	0.0266091	-0.0028342	0.0058675*	0.7197319*	-7.9214722		
	(0.0321692)	(0.002702)	(0.0014916)	(0.0488337)			
Merton	-0.0120413	0.0	0.2780027*	0.0	-114.24405	212.65	2
	(0.0096906)		(0.0068523)			(< 0.0001)	
CKLS	-0.0120413	0.0	0.2780027*	0.0	-114.24405	212.65	2
	(0.0096906)		(0.0068523)			(< 0.0001)	
Vasicek	0.0263404	-0.0027892	0.27861*	0.0	-113.7454	211.65	1
	(0.039679)	(0.002796)	(0.0069112)			(< 0.0001)	
CKLS	0.0263036	-0.0027853	0.2778343*	0.0	-113.7454	211.65	1
	(0.0395881)	(0.0027882)	(0.0068481)			(< 0.0001)	
CIR SR	0.0239849	-0.0026202	0.0185936*	0.5	-18.052057	20.26	1
	(0.0333743)	(0.0026638)	(0.000461)			(< 0.0001)	
CKLS	0.0239535	-0.0026167	0.0185449*	0.5	-18.052057	20.26	1
	(0.0332996)	(0.0026568)	(0.0004571)			(< 0.0001)	
Dothan	0.0	0.0	0.0014024*	1.0	-25.072527	34.30	3
			(0.0000346)			(< 0.0001)	
CKLS	0.0	0.0	0.0014024*	1.0	-25.072527	34.30	3
			(0.0000346)			(< 0.0001)	
GBM	0.0	-0.0005198	0.0014025*	1.0	-24.71264	33.58	2
		(0.0006127)	(0.0000346)			(< 0.0001)	
CKLS	0.0	-0.0005197	0.0014018*	1.0	-24.71264	33.58	2
		(0.0006124)	(0.0000346)			(< 0.0001)	
Brennan-Schwartz	0.0328208	-0.0033736	0.0014056*	1.0	-24.18784	32.53	1
	(0.0320716)	(0.0028554)	(0.000349)			(< 0.0001)	
CKLS	0.0327655	-0.0033679	0.0014009*	1.0	-24.18784	32.53	1
	(0.0319718)	(0.0028458)	(0.0000345)			(< 0.0001)	
CIR VR	0.0	0.0	0.0001193*	1.5	-129.09008	242.34	3
			(0.00000294)			(< 0.0001)	
CKLS	0.0	0.0	0.0001193*	1.5	-129.09008	242.34	3
			(0.00000294)			(< 0.0001)	
CEV	0.0	-0.0006618	0.0059095*	0.7185646*	-8.2633513	0.68	1
		(0.000632)	(0.0015005)	(0.0487707)		(0.4083)	
CKLS	0.0	-0.0006615	0.0059056*	0.7185646*	-8.2633513	0.68	1
		(0.0006315)	(0.0014993)	(0.0487705)		(0.4083)	

6.3 Results for the period under inflation targeting

The results, shown in Table 5, change substantially for the period after the adoption of inflation targeting (457 observations, from 4 February 2002 to 18 July 2011). Comparing the maximized Gaussian log likelihood values of the nested models to the same value for the unrestricted model, the Brennan-Schwartz model performs best, followed very closely by the CEV model, the Geometric Brownian Motion used by Black and Scholes (1976), and the Dothan model. The differences in log likelihoods between these four models are practically zero; and they all perform well. The χ^2 likelihood ratio tests confirm this. Under the null hypothesis that the nested model restrictions imposed are valid, we now fail to reject the same group of four models, with statistical rejection of the Merton, Vasicek, CIR (1985) and CIR (1980) models.

Table 5

Gaussian Estimates of Continuous-Time Models of the Short-Rate Under inflation targeting, 02/2002 - 07/2011

Model	α	β	σ^2	γ	Log Likelihood	χ^2 Test	df
Unrestricted	-0.0084862	0.0004217	0.0018747*	0.9932485*	285.74889		
	(0.0253701)	(0.0035477)	(0.000694)	(0.0873716)			
CKLS	-0.0084881	0.0004218	0.0018754*	0.9932581*	285.74889		
	(0.0253902)	(0.0035492)	(0.000694)	(0.0873659)			
Merton	-0.0053756	0.0	0.1434664*	0.0	222.677	126.14	2
	(0.006951)		(0.0049151)			(< 0.0001)	
CKLS	-0.0053756	0.0	0.1434664*	0.0	222.677	126.14	2
	(0.006951)		(0.0049151)			(< 0.0001)	
Vasicek	-0.0036918	-0.0001988	0.1434943*	0.0	222.67871	126.14	1
	(0.0295928)	(0.0033955)	(0.0049401)			(< 0.0001)	
CKLS	-0.036914	-0.0001987	0.01434658*	0.0	222.67871	126.14	1
	(0.0295959)	(0.0033948)	(0.0049151)			(< 0.0001)	
CIR SR	-0.0055954	0.0000328	0.015589*	0.5	269.82843	31.84	1
	(0.0264817)	(0.0033892)	(0.0005367)			(< 0.0001)	
CKLS	-0.0055955	0.0000328	0.0155895*	0.5	269.82843	31.84	1
	(0.0264914)	(0.0033893)	(0.0005341)			(< 0.0001)	
Dothan	0.0	0.0	0.0018256*	1.0	285.1108	1.28	3
			(0.0000625)			(0.7348)	
CKLS	0.0	0.0	0.0018256*	1.0	285.1108	1.28	3
			(0.0000625)			(0.7348)	
GBM	0.0	-0.0007434	0.0018245*	1.0	285.68935	0.12	2
		(0.0006909)	(0.0000625)			(0.9422)	
CKLS	0.0	-0.0007432	0.0018231*	1.0	285.68935	0.12	2
		(0.0006904)	(0.0000625)			(0.9422)	
Brennan-Schwartz	-0.0085331	0.0004283	0.0018221*	1.0	285.74591	0.01	1
	(0.0253553)	(0.0035494)	(0.0000628)			(0.9384)	
CKLS	-0.0085349	0.0004284	0.0018229*	1.0	285.74591	0.01	1
	(0.0253756)	(0.0035509)	(0.0000625)			(0.9384)	
CIR VR	0.0	0.0	0.0002305*	1.5	268.26711	34.96	3
			(0.0000079)			(< 0.0001)	
CKLS	0.0	0.0	0.0002305*	1.5	268.26711	34.96	3
			(0.0000079)			(< 0.0001)	
CEV	0.0	-0.0007421	0.001883*	0.9925148*	285.69302	0.11	1
		(0.0006921)	(0.0006965)	(0.0873336)		(0.7382)	
CKLS	0.0	-0.0007418	0.0018816*	0.992515*	285.69302	0.11	1
		(0.0006916)	(0.000696)	(0.0873318)		(0.7382)	

What the four best performing models have in common is the form of the diffusion function. The Brennan-Schwartz, GBM and Dothan models restrict γ to one. And the CEV model produces an estimate of the parameter γ of 0.9925 for the period. (The unrestricted model estimates γ at 0.9932, and both these estimates are statistically significant.)[8] The worst performing models, namely Merton and Vasicek, restrict γ to zero (i.e., no level effect). The CIR (1985) model implies a low volatility-on-level effect ($\gamma = 0.5$), and performs slighly better than the latter two. The CIR (1980)

[8]Nowman (1998) finds support for the same set of models for Japan and the United States.

restricts γ to 1.5, implying a higher sensitivity of volatility to level than any of the other models with a restriction on γ. Although it performs better than Merton, Vasicek, and CIR (1985), it is also statistically rejected.

The findings are strongly indicative of a level effect for South Africa: the conditional volatility is dependent on the level of the interest rate, with a γ coefficient approximately equal to one. This is a higher level of dependence (of interest rate conditional volatility on the interest rate level) than that found for the period prior to inflation targeting. The estimates of the same parameter for a set of advanced economies in Tse (1995), range from -0.36 to 1.73 (obtained using the Generalised Method of Moments), with "medium sensitivity" cases (defined somewhat arbitrarily) between 0 and 1.5. So, compared to the international evidence in Tse (1995), the sensitivity of interest rate volatility to its level for South Africa falls neatly within the medium sensitivity category. The next table contrasts unrestricted estimates of γ for different countries, obtained through maximum likelihood, and therefore more directly comparable.

Table 6

Comparison of International Estimates (Likelihood) of γ

Country	γ	Source
Australia	1.14	Brailsford and Maheswaran (1998)
Canada	0.44	McManus and Watt (1999)
France	2.83	Nowman (1998)
Italy	2.20	Nowman (1998)
Japan	0.98	Nowman (1998)
United Kingdom	0.28	Nowman (1997)
United States	1.36	Nowman (1997)

The likelihood estimates of γ vary from 0.28 for the United Kingdom, to 2.83 for France. Our findings for South Africa for the sub-sample under inflation targeting are closest to Nowman's (1998) findings for Japan, using the same estimation methods $-$ both in terms of valid models, and the nearly identical estimate of γ.

There is no significant evidence of mean reversion. Most models produce negative estimates of the β coefficient, but these are extremely close to zero. The same applies to α. The α and β parameters are not statistically significant in the unrestricted model. Lastly, observe that, again, the asymptotic bias resulting from the CKLS approximation is extremely small, and the estimates are almost identical for the CKLS approximation and the discrete model used by Nowman.

6.4 Discussion

The findings on the drift function parameters are consistent with the fact that the four best performing models have totally different implications regarding the drift function: Dothan has no drift; GBM and CEV imply constant proportional changes and no mean-reversion; while the Brennan-Schwartz implies reversion to the mean.

The weak evidence of mean reversion, and the associated observation that the best performing models have very different drift functions, is in a sense favourable. Estimates of the drift parameters under the objective probability measure (i.e., obtained through statistical analysis of market data) cannot be used directly for pricing - we need a valid estimate of the risk premium in the bond market. (This does not apply to the diffusion parameters however − see section 2.) Finding that differences in the specification of the drift function have little impact on model performance is therefore convenient. Second, as observed by CKLS, mean reverting drift functions tend to make term structure models more complex to handle in no-arbitrage analysis. Our findings suggest that, for South Africa, as found for the United States by CKLS, models with simple specifications of the drift function may still perform reasonably well, even if somewhat naïve, provided that the specification of the diffusion function is a realistic reflection of the relationship between interest rate volatility and the level of the interest rate. This relationship is clearly a central feature of short-rate dynamics, in South Africa and elsewhere.

7 Conclusion

Diffusion models which allow the conditional interest rate volatility to be moderately dependent on the interest rate level provide the best empirical fit for South African data. The constant elasticity of variance model, which imposes no quantitative restrictions on the sensitivity of interest rate volatility to the interest rate level, is the only model that fits the data in both sub-samples. Over the more recent sub-sample, after the adoption of inflation targeting, we find support for three well-known models where the magnitude of this sensitivity equals one, and for the constant elasticity of variance model, which estimates it at approximately one − a moderate level-effect, compared to available international evidence. We find no statistically significant evidence of a mean-reversion effect or a linear trend, at the weekly frequency.

Our findings are of practical use for the valuation of short-dated interest-rate derivatives, in applications where assuming that one factor, namely the short-rate, is the only state variable determining the current yield curve, is not exceedingly simplistic, and can be justified by practical implementation concerns. Of course, this will not be the case in many applications, and investigating multi-factor models for the South African term-structure is a natural extension of the present contribution. Our results will be relevant for any multi-factor model of the term-structure with embedded assumptions about the stochastic behavior of the South African short-rate.

The no arbitrage approach to interest rate modelling, which is now standard in mathematical finance and modern financial economics, was developed largely separately from monetary and macro-economics. Advances incorporating no-arbitrage restrictions in monetary policy models, or enriching arbitrage-free models with explicit treatments of monetary policy, are a very promising and challenging area for future research.[9]

8 Appendix

This derivation is standard, and follows Björk (2004), and Demange and Rochet (2005) closely. It is included in an attempt to keep the paper relatively self-contained. Applying Itô's formula to F^T gives:

$$dF^T = \left[\Lambda_r F^T\right] dt + \sigma_t \frac{\partial F^T}{\partial r} dW, \tag{16}$$

where Λ_r is the Dynkin operator of r, defined by $\Lambda_r = \frac{\partial}{\partial t} + \frac{\partial}{\partial r}\mu_t + \frac{1}{2}\frac{\partial^2}{\partial r^2}\sigma_t^2$, given $dr_t = \mu_t dt + \sigma_t dW_t$, and applied to F^T. Let $\alpha^T = \Lambda_r F^T$ and $\sigma^T = \sigma_t \frac{\partial F^T}{\partial s}$, so dF^T becomes $dF^T = \alpha^T dt + \sigma^T dW$. Similarly, Itô's formula applied to F^S gives

$$\begin{aligned} dF^S &= \left[\Lambda_r F^S\right] dt + \sigma_t \frac{\partial F^S}{\partial r} dW && (17) \\ &= \alpha^S dt + \sigma^S dW, && (18) \end{aligned}$$

where Λ_r is the Dynkin of r, given $dr_t = \mu_t dt + \sigma_t dW_t$, applied to F^S.

Consider a portfolio of the T- and S-bonds, with weights (δ_T, δ_S), and associated dynamics:

$$dV = V\left[\delta_T\left(\frac{dF^T}{F^T}\right) + \delta_S\left(\frac{dF^S}{F^S}\right)\right], \tag{19}$$

[9] See for example Ang, Dong, and Piazzesi (2005).

or

$$\frac{dV}{V} = \delta_T \left(\widetilde{\alpha}_T dt + \widetilde{\sigma}_T dW \right) + \delta_S \left(\widetilde{\alpha}_S dt + \widetilde{\sigma}_S dW \right), \tag{20}$$

where $\widetilde{\alpha}_T = \frac{\alpha^T}{F^T}$, and $\widetilde{\sigma}_T = \frac{\sigma^T}{F^T}$. Re-arranging,

$$\frac{dV}{V} = \left(\delta_T \widetilde{\alpha}_T + \delta_S \widetilde{\alpha}_S \right) dt + \left(\delta_T \widetilde{\sigma}_T + \delta_S \widetilde{\sigma}_S \right) dW. \tag{21}$$

The portfolio will have deterministic dynamics (i.e. no stochastic term) if $\delta_T \widetilde{\sigma}_T + \delta_S \widetilde{\sigma}_S = 0$. Hence choose (δ_T, δ_S) as $\delta_T = \frac{\widetilde{\sigma}_S}{\widetilde{\sigma}_S - \widetilde{\sigma}_T}$, and set $\delta_S = 1 - \delta_T = \frac{\widetilde{\sigma}_T}{\widetilde{\sigma}_T - \widetilde{\sigma}_S}$. Now substitute back into $\frac{dV}{V}$, to find

$$\frac{dV}{V} = \left(\frac{\widetilde{\alpha}_S \widetilde{\sigma}_T - \widetilde{\alpha}_T \widetilde{\sigma}_S}{\widetilde{\sigma}_T - \widetilde{\sigma}_S} \right) dt, \tag{22}$$

which has no stochastic term. Hence we constructed a risk-free portfolio (i.e. a portfolio with deterministic dynamics). No-arbitrage equilibrium then requires that the portfolio earns the risk-free rate, $\frac{dV}{V} = rdt$, or $\left(\widetilde{\alpha}_S \widetilde{\sigma}_T - \widetilde{\alpha}_T \widetilde{\sigma}_S \right) / \left(\widetilde{\sigma}_T - \widetilde{\sigma}_S \right) = r$ which, upon re-arranging, implies $\left(\widetilde{\alpha}_S - r \right) / \widetilde{\sigma}_S = \left(\widetilde{\alpha}_T - r \right) / \widetilde{\sigma}_T = \lambda$ – i.e., all bonds, of all maturities, will command the same ratio of excess return (relative to the risk-free rate) to volatility in equilibrium. This ratio represents the risk premium in the bond market, which determines the martingale measure – formally, through the Girsanov kernel.

Re-arranging gives

$$\begin{aligned} \Lambda_r F^T - r F^T &= 0 & (23) \\ F^T(T, r) &= 1, & (24) \end{aligned}$$

where Λ_r is the Dynkin of r, for r given by $dr_t = \left(\mu_t - \lambda_t \sigma_t \right) dt + \sigma_t dW_t$. This is the no-arbitrage term-structure equation. When solved for zero-coupon bonds, it gives us a no-arbitrage yield curve, associated with a given short-rate dynamics. Arbitrage-free prices of other interest rate derivatives are obtained by specifying the terminal condition, on $F^T(T, r)$, to reflect the derivative's payoff at maturity.

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